A MULTIPLICITY RESULT FOR A GENERALIZED PENDULUM EQUATION

Boris Rudolf

Department of Mathematics, Faculty of Electrical engineering STU,
812 19 Bratislava, Slovakia

We consider the second order differential equation

$$x'' + cx' = f(t, x, x') \tag{1}$$

with the periodic boundary conditions

$$x(a) = x(b), \qquad x'(a) = x'(b) \tag{2}$$

where $f : I = [a, b] \times R^2 \to R$ is a continuous bounded function, $c \neq 0$.

Our goal is to prove two types of existence results for the problem (1), (2) using a method of lower and upper solutions similarly as it is done in [4], [5] for the equation $x'' = f(t, x, x')$ with a bounded nonlinearity.

Rachůnková in [3] has proved the existence results with one sided linear growth conditions on f for the case when a lower solution is greater then an upper one. This result can be applied on (1), (2) when $|c| < \frac{1}{b-a}$. Our result need no restriction on a constant c.

Theorem 1 handle with the case when a lower solution is less then an upper one and is proved already in a more general situation when $f - cx'$ satisfies a Nagumo–Bernstein condition [1], [4]. We give our proof only for completeness.

We apply the existence results given in Theorems 1 and 2 to prove a multiplicity result for a problem (1), (2) with a periodic nonlinearity.

Definition. The function $\alpha(t)$ is called a lower solution for the problem (1), (2) if

$$\begin{aligned} \alpha''(t) + c\alpha'(t) &\geq f(t, \alpha(t), \alpha'(t)), \\ \alpha(a) = \alpha(b) \qquad & \alpha'(a) = \alpha'(b). \end{aligned}$$

Similarly the function $\beta(t)$ is called an upper solution for the problem (1), (2) if

$$\begin{aligned} \beta''(t) + c\beta'(t) &\leq f(t, \beta(t), \beta'(t)), \\ \beta(a) = \beta(b) \qquad & \beta'(a) = \beta'(b). \end{aligned}$$

If the strict inequalities hold α, β are called strict lower and upper solutions.

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Lemma 1. *Let α, β be a strict lower and upper solutions and $u(t)$ be a solution of the problem (1), (2).*

Then $\alpha(t) \leq u(t)$ implies $\alpha(t) < u(t)$ and $\beta(t) \geq u(t)$ implies $\beta(t) > u(t)$.

Proof. Let $\beta(t) \geq u(t)$ and $0 = u(t_0) - \beta(t_0)$ at $t_0 \in (a, b)$. Then

$$\begin{aligned} 0 \geq u(t_0)'' - \beta(t_0)'' =& u(t_0)'' - \beta(t_0)'' + cu(t_0)' - c\beta(t_0)' = \\ =& f(t_0, u(t_0), u'(t_0)) - \beta(t_0)'' - c\beta(t_0)' \geq \\ \geq& f(t_0, \beta(t_0), \beta'(t_0)) - \beta(t_0)'' - c\beta(t_0)' > 0, \end{aligned}$$

a contradiction.

Let $0 = u(a) - \beta(a)$, $u(t) < \beta(t)$ for $t \in (a, b)$. Then $u'(a) = \beta'(a)$ and we obtain the same contradiction as above. □

Let $X = C^1(I)$, $domL = \{x(t) \in C^2(I),\ x \text{ satisfies } (2)\}$, $Z = C(I)$. We denote

$$L : domL \subset X \to Z, \qquad Lx = x'' + cx',$$

$$N : X \to Z, \qquad Nx(t) = f(t, x(t), x'(t)).$$

The problem (1), (2) is equivalent to the operator equation

$$Lx = Nx,$$

where the operator N is L-compact [1].

We denote

$$\Omega_{r,\rho} = \{x(t) \in C^1(I), \quad ||x|| < r, \quad ||x' + cx|| < \rho\}.$$

Lemma 2. *Let*

(i) there is a constant $r > 0$ such that $f(t, r, 0) > 0$ and $f(t, -r, 0) < 0$,

(ii) $|f(t, x, y)| \leq M$,

Then there is $\rho_0 > 0$ such that the topological degree

$$D(L, N, \Omega_{r,\rho}) = 1 \qquad (mod\ 2)$$

for each $\rho > \rho_0$ i.e. there is a solution $x(t)$ of (1), (2) such that $|x(t)| < r$, $|x'(t) + cx(t)| < \rho$.

Proof.

We consider the homotopy

$$Lx = \tilde{N}(x, \lambda)$$

defined by the parametric system of equations

$$x'' + cx' = \lambda f(t, x, y) + (1 - \lambda)x, \tag{6}$$

$$x(a) = x(b) \qquad x'(a) = x'(b). \tag{2}$$

Now $-r$, r are a strict lower and upper solutions of the problem (6).

As $|\lambda f(t, x, y) + (1 - \lambda)x,| \leq M + r$, then for each solution of (6) such that $|x(t)| \leq r$ there is $|x'(t) + cx(t)| \leq \dfrac{b-a}{2}(M + r) = \rho_0$.

The above estimation and Lemma 1 imply that no solution of (6), (2) lies on the boundary of $\partial\Omega_{r,\rho}$, $\rho \geq \rho_0$.

By the generalized Borsuk theorem [2]

$$D(L, \tilde{N}(., 1), \Omega_{r,\rho}) = D(L, \tilde{N}(., 0), \Omega_{r,\rho}) = 1 \qquad (mod\ 2)$$

and Lemma 2 is proved. □

Theorem 1. *Let*

(i) $\alpha(t) < \beta(t)$ be a strict lower and upper solutions of the problem (1), (2).

(ii) $|f(t,x,y)| \leq M$, for each (t,x,y), $t \in I$ $\alpha(t) \leq x \leq \beta(t)$, $y \in R$.

Then there is a constant ρ_0 such that for each $\Omega_1 = \{x(t) \in C^1(I), \quad \alpha(t) < x(t) < \beta(t), \quad ||x' + cx|| < \rho\}$, $\rho > \rho_0$ there is

$$D(L, N, \Omega_1) = 1 \qquad (mod\ 2)$$

i.e. there is a solution $x(t) \in \Omega$ of (1), (2).

Proof.

Let $r = max\{||\alpha||, ||\beta||\}$,

We define a perturbation

$$f^*(t,x,y) = \begin{cases} f(t,\beta(t),y) + M(r - \beta(t)) + M & x > r+1, \\ f(t,\beta(t),y) + M(x - \beta(t)) & \beta(t) < x \leq r+1, \\ f(t,x,y) & \alpha(t) \leq x\beta(t), \\ f(t,\alpha(t),y) - M(\alpha(t) - x) & -r-1 \leq x < \alpha(t), \\ f(t,\alpha(t),y) - M - M(\alpha(t) + r) & x < -r-1. \end{cases}$$

Then $|f^*| \leq 2M$ and the assumptions of Lemma 2 are satisfied for $\Omega_{r+1,\rho}$, $\rho > \rho_0$ where $\rho_0 = \dfrac{b-a}{2}(2M + r + 1)$.

Suppose $u(t) \in \Omega_{r+1,\rho}$ is a solution of the problem

$$x'' + cx' = f^*(t,x,x'), \quad (2), \tag{7}$$

$$x(a) = x(b) \qquad x'(a) = x'(b). \tag{2}$$

We show that $\alpha \leq u \leq \beta$.

Let $v(t) = u(t) - \beta(t)$ attains its maximum $v_{max} > 0$. Then $\beta(t) + v_{max}$ is a strict upper solution of (7), (2). Lemma 1 implies $u(t) < \beta(t) + v_{max}$ a contradiction.

That means $u(t)$ is a solution of (1), (2).

Then

$$D(L, N^*, \Omega_{r+1,\rho}) = D(L, N^*, \Omega_1) = D(L, N, \Omega_1) = 1 \qquad (mod\ 2). \quad \square$$

Now we assume that a lower and upper solutions are in a more general position.

Theorem 2. *Let*

(i) $|f(t,x,y)| < M$,

(ii) α, β, $\alpha(t) \not\leq \beta(t)$, be a strict lower and upper solutions for the problem (1), (2).

Then there are constants $r, \rho_0 > 0$ such that

$$D(L, N, \Omega_2) = 1 \qquad (mod\ 2)$$

where $\Omega_2 = \{x(t) \in C^1(I),\ \exists t_\alpha, t_\beta \in I,\ \beta(t_\beta) < x(t_\beta),\ x(t_\alpha < \alpha(t_\alpha),\ ||x|| < r,\ ||x' + cx|| < \rho\} \qquad \rho > \rho_0$, *i.e. there is a solution $x(t) \in \Omega_2$ of the problem (1), (2).*

Proof. Let $r = \max(||\alpha||, ||\beta||) + \dfrac{(b-a)}{c} M$.

We define a perturbation f^* by

$$f^*(t,x,y) = \begin{cases} f(t,x,y) + M & x > r+1, \\ f(t,x,y) + M(x-r) & r < x \le r+1, \\ f(t,x,y) & -r \le x \le r, \\ f(t,x,y) + M(x+r) & -r-1 \le x < -r, \\ f(t,x,y) - M & x < -r-1. \end{cases}$$

Clearly $r+1$, $-r-1$ are a strict lower and upper solutions of the problem

$$x'' + cx' = f^*(t,x,x'), \tag{8}$$

$$x(a) = x(b) \qquad x'(a) = x'(b). \tag{2}$$

As $|f^*| < 2M$ then for each solution of (8) the boundary conditions (2) imply $|x'(t) + cx(t)| \le (b-a)M$. Then $max|x(t)| \le \dfrac{(b-a)M}{c}$.

Set $\rho_0 = \dfrac{(b-a)}{2}(2M + r + 1)$.

Then for $\rho > \rho_0$

$$D(L, N^*, \Omega_{r+1,\rho}) = 1 \qquad (mod\ 2)$$

Let now

$$\Omega_l = \{x(t) \in \Omega_{r+1,\rho}, \quad -r-1 < x < \beta\},$$
$$\Omega_u = \{x(t) \in \Omega_{r+1,\rho}, \quad \alpha < x < r+1\}.$$

Then

$$D(L, N^*, \Omega_l) = D(L, N^*, \Omega_u) = 1 \qquad (mod\ 2)$$

Set $\Omega_m = \Omega_{r+1,\rho} \setminus \left(\overline{\Omega_l \cup \Omega_u}\right)$.

As $-r-1$, α, $r+1$, β are strict lower and upper solutions, Lemma 1 implies there is no solution $u \in \partial\Omega_m$.

The addition property of the degree means

$$D(L, N^*, \Omega_m) = 1 \qquad (mod\ 2)$$

and finally the excision property implies

$$D(L, N^*, \Omega_m) = D(L, N^*, \Omega_2) = D(L, N, \Omega_2) = 1 \qquad (mod\ 2). \quad \square$$

We apply the previous results to a periodic boundary value problem for a generalized oscillator

$$x'' + cx' = f(t, x, x') \tag{1}$$

$$x(a) = x(b) \qquad x'(a) = x'(b), \tag{2}$$

assuming that the function f is 2–π periodic in variable x.

Assume that there are $\alpha(t)$, $\beta(t)$ a strict lower and upper solutions of (1), (2). The periodicity of f implies that $\alpha(t) + 2k\pi$, $\beta(t) + 2k\pi$ are again a strict lower and upper solutions of (1), (2) for each $k \in Z$.

Then there is a $k \in Z$ such that $\alpha(t) + 2k\pi < \beta(t)$ and $\alpha(t) + (2k + 1)\pi \nleq \beta(t)$. Then Theorem 1 and Theorem 2 imply there are two different families $x_1(t) + 2k\pi$, $x_2(t) + 2k\pi$ of solutions of the problem (1), (2).

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