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ORDER-TOPOLOGICAL LATTICE EFFECT ALGEBRAS

ZDENKA RIEČANOVÁ

Abstract. We study order convergence of nets in lattice effect algebras, which generalized orthomodular lattices, including Boolean algebras and MV-algebras in quantum or fuzzy probability theory. We show that in a complete atomic (o)-continuous effect algebra E the order convergence of nets is topological if and only if the order topology on E is Hausdorff. If moreover E is distributive (e.g., MV-algebra) then the order topology is compact Hausdorff.

1. Introduction and basic definitions

Effect algebras, or equivalent in some sense D-posets were introduced as carriers of probability measure in quantum or fuzzy probability theory. Elements of these structures represent quantum effects or fuzzy events that may be unsharp or imprecise ([6], [13]). Lattice ordered effect algebras generalize orthomodular lattices [12] including Boolean algebras and MV-algebras [1], [2], [10], [11], [14].

Definition 1. [6]. *A structure $(E; \oplus, 0, 1)$ is called an effect-algebra if 0, 1 are two distinguished elements and $\oplus$ is a partially defined binary operation on P which satisfies the following conditions for any $a, b, c \in E$:*

(i) $b \oplus a = a \oplus b$ *if* $a \oplus b$ *is defined,*
(ii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ *if one side is defined,*
(iii) *for every* $a \in P$ *there exists a unique* $b \in P$ *such that* $a \oplus b = 1$ *(we put* $a' = b$*),*
(iv) *if* $1 \oplus a$ *is defined then* $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. In every effect algebra E we can define the partial operation $\ominus$ and the partial order $\leq$ by putting

$$a \leq b \text{ and } b \ominus a = c \text{ iff } a \oplus c \text{ is defined and } a \oplus c = b.$$

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Since $a \oplus c = a \oplus d$ implies $c = d$, the $\ominus$ and the $\leq$ are well defined. If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete effect algebra*). If $(E; \oplus, 0, 1)$ is an effect algebra then $(E; \ominus, 0, 1)$ with the partial binary operation $\ominus$ defined above is a *D-poset*, introduced by Kôpka [13] as a new algebraic structure of fuzzy sets, and vice versa, [14]. For more details on D-posets and efefct algebras we refer the reader to [3], [15].

Definition 2. *Elements a and b of a lattice effect algebra E are called compatible (written $a \leftrightarrow b$) if $a \vee b = a \oplus (b \ominus (a \wedge b))$,* [14].

On many places we will need the following statement proved in [11].

Lemma 3. *Let E be a lattice effect algebra and $A \subseteq E$ with $\bigvee A$ existing in E. If $b \in E$ is compatible with every $a \in A$ then $b \leftrightarrow \bigvee A$ and $b \wedge (\bigvee A) = \bigvee\{a \wedge b \mid a \in E\}$.*

A lattice effect algebra is called *modular* or *distributive* if E as a lattice has these properties [9]. A lattice effect algebra is called an *MV-effect algebra* if every two elements $a, b \in E$ are compatible. It has been shown by Kôpka and Chovanec [14] that an MV-effect algebra E can be organized into an MV-algebra and vice versa.

2. Topological Effect Algebras

Assume that $(\mathcal{E}; \prec)$ is a directed set and $(P; \leq)$ is a poset. A net of elements of P is denoted by $(a_\alpha)_{\alpha \in \mathcal{E}}$. If $a_\alpha \leq a_\beta$ for all $\alpha, \beta \in \mathcal{E}$ such that $\alpha \prec \beta$ then we write $a_\alpha \uparrow$. If moreover $a = \bigvee\{a_\alpha \mid \alpha \in \mathcal{E}\}$ we write $a_\alpha \uparrow a$. The meaning of $a_\alpha \downarrow$ and $a_\alpha \downarrow a$ is dual. For instance, $a \uparrow u_\alpha \leq v_\alpha \downarrow b$ means that $u_\alpha \leq v_\alpha$ for all $\alpha \in \mathcal{E}$ and $u_\alpha \uparrow a$ and $v_\alpha \downarrow b$. We will write $b \leq a_\alpha \uparrow a$ if $b \leq a_\alpha$ for all $\alpha \in \mathcal{E}$ and $a_\alpha \uparrow a$.

A net $(a_\alpha)_{\alpha \in \mathcal{E}}$ of elements of a poset $(P; \leq)$ *order converges* ((o)-converges, for short) to a point $a \in P$ if there are nets $(u_\alpha)_{\alpha \in \mathcal{E}}$ and $(v_\alpha)_{\alpha \in \mathcal{E}}$ of elements of P such that

$$a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a\,.$$

We write $a_\alpha \stackrel{(o)}{\to} a$ *in* P (or briefly $a_\alpha \stackrel{(o)}{\to} a$).

The strongest (biggest) topology on a poset $(P; \leq)$ such that (o)-convergence of nets of elements of P implies topological convergence is called *order topology* ((o)-topology) on P and it is denoted by τ_o. The *order sequence topology* denoted by τ_{os} is the strongest topology on P such that (o)-convergence of sequences implies topological convergence. We can show that $F \subseteq P$ is τ_o-closed (τ_{os}-closed) set iff F includes (o)-limits of all order convergent nets (sequences) of elements of F. In spite of that, the (o)-convergence and τ_o-convergence of nets in (even complete) lattices need not coincide. Moreover, the fact that in a lattice L the order convergence of filters is topological does not imply the same statement for order convergence of nets as we have shown in [24].

For complete orthomodular lattices (including Boolean algebras) it has been shown in [24] and [4] that they are (o)-topological, i.e., order continuous lattices in which the order convergence of nets is identical with their convergence in the order topology iff they are atomic and (o)-continuous lattices.

We will be concerned with the above mentioned problem for complete atomic effect algebras and *net-theoretical convergences*, because convergences of nets play important task in the probability (or measure) theory on these structures [23].

Recall that an arbitrary system $G = (a_\kappa)_{\kappa \in H}$ of not necessarily different elements of an effect algebra E is called $\oplus$-*orthogonal* if for every finite set $K \subseteq H$ the element $\bigoplus\{a_\kappa \mid \kappa \in K\}$ exists in E. If $\bigvee\{\bigoplus_{\kappa \in K} a_\kappa \mid K \subseteq H$ is finite$\}$ exists then we put $\bigoplus_{\kappa \in H} a_\kappa = \bigvee\{\bigoplus_{\kappa \in K} a_\kappa \mid K \subseteq H$ is finite$\}$. An Archimedean effect algebra E is called *separable* if every $\oplus$-orthogonal systems of elements of E is at most countable. More detailed these notions are discussed in [25].

Lemma 4. *$\tau_o = \tau_{os}$ on every complete separable effect algebra E.*

Proof. In view of definitions of τ_o and τ_{os} we have $\tau_o \subseteq \tau_{os}$, as for every sequence $(x_n)_{n=1}^{\infty}$ we have $x_n \stackrel{(o)}{\to} x$ implies $x_n \stackrel{\tau_o}{\to} x$. Let $F \subseteq E$ be a τ_{os}-closed set and $(x_\alpha)_{\alpha \in \mathcal{E}}$ be a net of elements of E such that $x_\alpha \stackrel{(o)}{\to} x \in E$. Since E is complete and separable, by [20, Theorem 4.7] there are $\alpha_1 \leq \alpha_2 \leq \dots$ in $\mathcal{E}$ such that $x_{\alpha_n} \stackrel{(o)}{\to} x$, hence $x \in F$ and F is τ_o-closed. It follows that $\tau_{os} \subseteq \tau_o$ and hence $\tau_o = \tau_{os}$. ☐

In the paper by an *order topological lattice* ((o)-topological, for short) we mean a lattice L whose order convergence of nets of elements coincides with convergence in the order topology τ_o and makes lattice operations continuous. For a lattice L a subset $D \subseteq L$ is called a *full sub-lattice* if for all $P, Q \subseteq D$ with $\bigvee P$ and $\bigwedge Q$ existing in L we have $\bigvee P, \bigwedge Q \in D$.

For net-theoretical convergence we will need some statements concerning the relativizations. Note that, in general, for a complete lattice L with order topology τ_o and its sublattice D with order topology τ_o^D, need not be $\tau_o^D = \tau_o \cap D$. Thus the fact that L is (o)-topological (in the sense of net convergences) does not imply that D is (o)-topological, even in the case when the convergence of filters is (o)-topological. All these facts have been shown in [24, Example 4.1].

Lemma 5. *Let D be a sublattice of a lattice L and τ_o^D and τ_o be order topologies on D and L, respectively. Let D be τ_o-closed. Then:*

(i) *D is a full sublattice of L.*

(ii) *If L is complete then D is complete as well and for $x_\alpha, x \in D$:*
$x_\alpha \stackrel{(o)}{\to} x$ *(in D) iff* $x_\alpha \stackrel{(o)}{\to} x$ *(in L).*
$x_\alpha \stackrel{\tau_o^D}{\to} x$ *(in D) iff* $x_\alpha \stackrel{\tau_o}{\to} x$ *(in L),*

(iii) *If L is complete and order topological then D is order topological.*

(iv) *Let L be complete and $f : D \to L$ be a map such that for $x_\alpha, x \in D$:*
$x_\alpha \stackrel{(o)}{\to} x$ *(in D)* $\implies f(x_\alpha) \stackrel{(o)}{\to} f(x)$ *(in L). Then for $y_\alpha, y \in D$:*
$y_\alpha \stackrel{\tau_o}{\to} y \implies f(y_\alpha) \stackrel{\tau_o}{\to} f(y)$.

Proof. (i) Assume that $A \subseteq D$ and $\bigvee A$ exists in E. Set $x_\alpha = \bigvee \alpha$, for all finite $\alpha \subseteq A$. Then $x_\alpha \in D$ and $x_\alpha \uparrow \bigvee A$, which gives $x_\alpha \stackrel{\tau_o}{\to} \bigvee A$, hence $\bigvee A \in D$. Dually, if $B \subseteq D$ and $\bigwedge B$ exists in L then $\bigwedge B \in D$.

(ii) As by (i) for each $H \subseteq D$ we have $\bigvee H, \bigwedge H \in D$ we obtain that for $x_\alpha \in D$, $\alpha \in \mathcal{E}$ we have

$$\bigvee_{\beta\in\mathcal{E}} \bigwedge_{\alpha\geq\beta} x_\alpha = \bigwedge_{\beta\in\mathcal{E}} \bigvee_{\alpha\geq\beta} x_\alpha \text{ (in } L) \text{ iff } \bigvee_{\beta\in\mathcal{E}} \bigwedge_{\alpha\geq\beta} x_\alpha = \bigwedge_{\beta\in\mathcal{E}} \bigvee_{\alpha\geq\beta} x_\alpha \text{ (in } D)$$

which is equivalent to

$$x_\alpha \stackrel{(o)}{\to} x \text{ (in } L) \text{ iff } x_\alpha \stackrel{(o)}{\to} x \text{ (in } D)$$

as D and L are complete lattices. It follows that $F \subseteq D$ is τ_o^D-closed iff F is τ_o-closed. Thus for $U \subseteq D$ we have $U \in \tau_o^D$ iff $L \setminus (D \setminus U) \in \tau_o$ and hence for $x_\alpha, x \in D$ we have $x_\alpha \stackrel{\tau_o^D}{\to} x$ iff $x_\alpha \stackrel{\tau_o}{\to} x$ iff $\tau_o^D = \tau_o \cap D$.

(iii) This is a consequence of (ii).

(iv) We have to prove that f is a continuous map of (D, τ_o^D) into (L, τ_o), since by (ii) for $y_\alpha, y \in D$ we have $y_\alpha \stackrel{\tau_o^D}{\to} y$ iff $y_\alpha \stackrel{\tau_o}{\to} y$. Assume that $F \subseteq L$ is τ_o-closed and $x_\alpha \in f^{-1}(F)$, $\alpha \in \mathcal{E}$. Then $f(x_\alpha) \in F$ and $x_\alpha \stackrel{(o)}{\to} x$ (in D) implies $f(x_\alpha) \stackrel{(o)}{\to} f(x)$ (in L) which gives $f(x) \in F$. Hence $x \in f^{-1}(F)$, which proves that $f^{-1}(F)$ is τ_o^D-closed. □

Recall that a lattice effect algebra E is (o)-*continuous* if for $x_\alpha, x, y \in E$: $x_\alpha \uparrow x \implies x_\alpha \wedge y \uparrow x \wedge y$, [8]. In every (o)-continuous effect algebra if $x_\alpha \stackrel{(o)}{\to} x$ and $y_\alpha \stackrel{(o)}{\to} y$ then $x_\alpha \vee y_\alpha \stackrel{(o)}{\to} x \vee y$ and $x_\alpha \wedge y_\alpha \stackrel{(o)}{\to} x \wedge y$.

Theorem 6. *In every complete (o)-continuous effect algebra E, for $x_\alpha, x, y \in E$:*

(i) $x_\alpha \stackrel{\tau_o}{\to} x \implies x_\alpha \vee y \stackrel{\tau_o}{\to} x \vee y$

(ii) $x_\alpha \stackrel{\tau_o}{\to} x \implies x_\alpha \wedge y \stackrel{\tau_o}{\to} x \wedge y$

(iii) $x_\alpha \stackrel{\tau_o}{\to} x \implies x'_\alpha \stackrel{\tau_o}{\to} x'$

Proof. (i)–(iii) follow from (o)-continuity of E using Lemma 5, (iv). □

Definition 7. *A complete effect algebra E is (o)-topological (order topological) if (o)-convergence of nets of elements coincides with τ_o-convergence and E is (o)-continuous.*

An element u of an effect algebra E is called *finite* if there is a finite sequence $\{p_1, \dots, p_n\}$ of not necessarily different atoms of E such that $u = p_1 \oplus p_2 \oplus \cdots \oplus p_n$. If E is complete then for every $x \in E$ we have $x = \bigvee\{u \in E \mid u \leq x$, u is finite$\}$, by [25, Theorem 3.1]. If E is complete atomic and (o)-continuous then the join of two finite elements is finite as well [21, Theorem 4.4].

Theorem 8. *A complete atomic (o)-continuous effect algebra E is (o)-topological iff τ_o on E is Hausdorff.*

Proof. (1) Assume that τ_o on E is Hausdorff and for $x, x_\alpha \in E$ let $x_\alpha \xrightarrow{\tau_o} x$, $\alpha \in \mathcal{E}$. If $a \in E$ is an atom such that $a \leq x$ then by Theorem 6 we have $x_\alpha \wedge a \xrightarrow{\tau_o} x \wedge a = a$. It follows that there is $\alpha_a \in \mathcal{E}$ such that $a \leq x_\alpha$ for all $\alpha \geq \alpha_a$, since otherwise there is a cofinal $\mathcal{E}' \subseteq \mathcal{E}$ such that $x_\alpha \wedge a \xrightarrow{\tau_o} 0$, $\alpha \in \mathcal{E}'$. By [16], we obtain that $x_\alpha \ominus a \xrightarrow{\tau_o} x \ominus a$, $\alpha \geq \alpha_a$. By the same argument, for an atom $b \leq x \ominus a$ there is $\alpha_b \geq \alpha_a$ such that for all $\alpha \geq \alpha_b$ we have $b \leq x_\alpha \ominus a$ which gives $a \oplus b \leq x_\alpha$. By induction, for every finite element $u = a_1 \oplus a_2 \oplus \cdots \oplus a_n \leq x$, where $a_1, \dots, a_n \in E$ are not necessary different atoms, there is $\alpha_u \in \mathcal{E}$ such that for all $\alpha \geq \alpha_u$ we have $u \leq x_\alpha$ and hence $u \leq \bigwedge_{\alpha \geq \alpha_u} x_\alpha$. We obtain that $x \leq \bigvee_{\beta \in \mathcal{E}} \bigwedge_{\alpha \geq \beta} x_\alpha$, because $x = \bigvee\{u \in E \mid u \leq x$, u is finite$\}$. Further, $x_\alpha \xrightarrow{\tau_o} x \implies x'_\alpha \xrightarrow{\tau_o} x'$, which gives $x' \leq \bigvee_{\beta \in \mathcal{E}} \bigwedge_{\alpha \geq \beta} x'_\alpha$. By D'Morgan laws we obtain $\bigwedge_{\beta \in \mathcal{E}} \bigvee_{\alpha \geq \beta} x_\alpha \leq x$. We conclude that $x = \bigvee_{\beta \in \mathcal{E}} \bigwedge_{\alpha \geq \beta} x_\alpha = \bigwedge_{\beta \in \mathcal{E}} \bigvee_{\alpha \geq \beta} x_\alpha$ which is equivalent to $x_\alpha \xrightarrow{(o)} x$, because

$$x \uparrow \bigwedge_{\alpha \geq \beta} x_\alpha \leq x_\alpha \leq \bigvee_{\alpha \geq \beta} x_\alpha \downarrow x\,.$$

(2) If E is (o)-topological then τ_o on E is Hausdorff, as the (o)-limit of an (o)-convergent net is unique. □

Theorem 9. *Let E be a complete atomic (o)-topological effect algebra. Then*

(i) *For every atom a of E the intervals $[a, 1]$ and $[0, a']$ are τ_o-clopen sets.*
(ii) *For every two finite elements $u, v \in E$ the intervals $[u, 1]$, $[0, u']$ and $[u, v']$ are τ_o-clopen sets.*
(iii) *Every $x \in E$ has a neighborhood base consisting of τ_o-clopen sets $[u, v']$, u, v finite.*

Proof. (i) Evidently, $[a, 1]$ and $[0, a']$ are τ_o-closed, since $\tau_i \subseteq \tau_o$. Let $x_\alpha \xrightarrow{\tau_o} x$, for $\alpha \in \mathcal{E}$ and $x \in [0, 1]$. By Theorem 6, $x_\alpha \wedge a \xrightarrow{\tau_o} x \wedge a = a$ and since τ_o is

Hausdorff, there is $\alpha_0 \in \mathcal{E}$ such that for all $\alpha \geq \alpha_0$ we have $x_\alpha \wedge a = a$, as otherwise $x_\alpha \wedge a \xrightarrow{\tau_o} 0$. It follows that $x_\alpha \in [a,1]$, for all $\alpha \geq \alpha_0$, which gives that $[a,1]$ is open and hence also $[0,a']$ is open because $x \in [a,1]$ iff $x' \in [0,a']$.

(ii) If $u = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ where a_k are atoms of E and $x_\alpha \xrightarrow{\tau_o} x \in [u,1]$ then $a_1 \wedge x_\alpha \xrightarrow{\tau_o} a_1 \wedge x = a_1$ and hence $a_1 \leq x_\alpha$ for all $\alpha \geq \alpha_1$. By [16, Theorem 3.3] for $\alpha \geq \alpha_1$ we have $x_\alpha \ominus a_1 \xrightarrow{\tau_o} x \ominus a_1$. Since $a_2 \leq x \ominus a_1$, there is $\alpha_2 \geq \alpha_1$ such that for $\alpha \geq \alpha_2$ we have $a_2 \leq x_\alpha \ominus a_1$ which gives $a_1 \oplus a_2 \leq x_\alpha$. By induction there is $\alpha_n \geq \alpha_k$, $k = 1, 2, \dots, n-1$ such that for all $\alpha \geq \alpha_n$ we have $u \leq x_\alpha$. This proves that $[u,1]$ is τ_o-clopen. It follows that $[0,u']$ is τ_o-clopen for every finite $u \in E$. Thus for all finite $u, v \in E$ we have $[u,v'] = [u,1] \cap [0,v']$ is τ_o-clopen.

(iii) Let $x \in E$ be arbitrary and $x \in \mathcal{U}(x) \in \tau_o$. Put $P_x = \{u \in E \mid u \leq x,$ u is finite$\}$ and $Q_{x'} = \{v \in E \mid v \leq x',\ v \text{ is finite}\}$. Then $x = \bigvee P_x$ and $x' = \bigvee Q_{x'}$. For every finite set $F \subseteq P_x \cup Q_{x'}$ we put $u_F = \bigvee(F \cap P_x)$ and $v_F = \bigvee(F \cap Q_{x'})$. Evidently $\mathcal{E} = \{F \subseteq P_x \cup Q_{x'} \mid F \text{ is finite}\}$ is directed by set inclusion and $u_F \uparrow x$, $v_F \uparrow x'$ which gives $v'_F \downarrow x$. Since $x \in \mathcal{U}(x) \in \tau_o$ there is $F_0 \in \mathcal{E}$ such that $[u_{F_0}, v'_{F_0}] \in \mathcal{U}(x)$ (see, e.g., Appendix B by H. Kirchheimová and Z. Riečanová, Proposition B.2.1 in [19]) and the interval $[u_{F_0}, v'_{F_0}]$ is τ_o-clopen, as u_{F_0} and v_{F_0} are finite elements of E ([21, Theorem 4.4]). □

A *direct product* of a family $\{E_\kappa \mid \kappa \in H\}$, $H \neq \emptyset$, of effect algebras is the effect algebra $(\widehat{E}; \hat{\oplus}, \hat{0}, \hat{1})$, where $\widehat{E} = \prod\{E_\kappa \mid \kappa \in H\}$ is a Cartesian product and all operations $\hat{\oplus}$, $\hat{0}$, $\hat{1}$ are defined componentwise. It follows that also the partial order and lattice operations (for lattice ordered E_κ, $\kappa \in H$) in $\widehat{E}$ are defined componentwise.

Recall that effect algebras $(E; \oplus_E, 0_E, 1_E)$ and $(F; \oplus_F, 0_F, 1_F)$ are *isomorphic* if there exists a bijective map $\varphi : E \to F$ such that

(i) $\varphi(1_E) = 1_F$,
(ii) for all $a, b \in E$: $a \leq b'$ iff $\varphi(a) \leq \varphi(b')$,
in which case $\varphi(a \oplus_E b) = \varphi(a) \oplus_F \varphi(b)$.

We write $E \cong F$. Sometimes we identify E with $\varphi(E)$.

In every complete effect algebra E the center $C(E) = \{z \in E \mid z \wedge z' = 0$ and $z \leftrightarrow x$ for all $x \in E\}$ is a complete Boolean algebra and $S(E) = \{z \in E \mid z \wedge z' = 0\}$ is a complete orthomodular lattice, and both are full sublattices of E, as we have shown in [11] and [25]. It follows that $C(E)$ and $S(E)$ are τ_o-closed subsets of E, since they evidently contain all (o)-limits of their (o)-convergent nets. In view of Lemma 5, (iii), $C(E)$ and $S(E)$ are (o)-topological. It follows that $C(E)$ and $S(E)$ are atomic and (o)-continuous [4, Lemma 2.2]. We have shown [18, Lemma 4.3] that then E can be decomposed into direct product of irreducible effect algebras.

Proposition 10. *In every (o)-topological effect algebra E:*

(i) *The center $C(E)$ is a complete atomic Boolean algebra.*
(ii) *The set of sharp elements $S(E)$ is a complete atomic and (o)-continuous orthomodular lattice.*
(iii) $E \cong \prod\{[0,p] \mid p$ *is an atom of* $C(E)\}$, $[0,p]$ *are irreducible effect algebras.*

Note that the atomicity of $C(E)$ and $S(E)$ for a complete effect algebra E does not imply that E is atomic.

Example 11. *Let $E = [0,1] \subseteq R$ with defined $a \oplus b = a+b$ iff $a+b \leq 1$. E is a complete MV-effect algebra, since every pair of elements of E are comparable and hence compatible. Clearly E has separated intervals which implies that τ_i on E is Hausdorff and hence $\tau_o = \tau_i$ is a compact Hausdorff topology on E (see [4] and [17]). Moreover, as E is complete, for $x_\alpha \in E$, $\alpha \in \mathcal{E}$ we have $x_\alpha \stackrel{(o)}{\to} x$ iff $x = \bigvee_{\beta\in\mathcal{E}} \bigwedge_{\alpha\geq\beta} x_\alpha = \bigwedge_{\beta\in\mathcal{E}} \bigvee_{\alpha\geq\beta} x_\alpha$ iff $x_\alpha \stackrel{\tau_o}{\to} x$, hence (o)-convergence is a topological convergence on E. Evidently E is (o)-continuous. Note that $C(E) = S(E) = \{0,1\}$.*

3. Compact (o)-topological effect algebras

Definition 12. *A complete effect algebra is compact (o)-topological if E is (o)-topological and τ_o is compact.*

The *interval topology* τ_i on a bounded lattice L is the topology for which an open base is generated by complements of finite unions of closed intervals. It follows that $\tau_i \subseteq \tau_o$. If τ_i is Hausdorff then $\tau_i = \tau_o$ (see [5]). By [7], τ_i on L is compact iff L is a complete lattice. Thus *on every complete effect algebra E with Hausdorff interval topology the order topology τ_o is compact and Hausdorff*. Nevertheless, such an effect algebra E need not be (o)-topological since E need not be (o)-continuous (see Example 20).

Definition 13. *A bounded lattice L has separated intervals, if given any two disjoint intervals $[a,b], [c,d] \subseteq L$, the lattice L can be covered by a finite number of closed intervals each of which is disjoint with at least one of the intervals $[a,b]$ and $[c,d]$,*

We have shown in [17, Lemma 2.2] that *the interval topology on a complete lattice L is Hausdorff iff L has separated intervals*.

Theorem 14. *Every complete atomic (o)-continuous effect algebra with separated intervals is compact (o)-topological.*

Proof. Since τ_i is Hausdorff, we have $\tau_o = \tau_i$ and hence τ_o is compact Hausdorff topology, as E is complete. By Theorem 8, E is (o)-topological.. □

Lemma 15. *Let $\{E_\kappa \mid \kappa \in H\}$ be a family of complete lattice effect algebras.*

(i) $\widehat{E} = \prod\{E_\kappa \mid \kappa \in H\}$ *has separated intervals iff for every* $\kappa \in H$, E_κ *has separated intervals.*

(ii) τ_i *on* $\widehat{E}$ *is Hausdorff iff* $\tau_i^{(\kappa)}$ *is Hausdorff for all* $\kappa \in H$.

Proof. For any $\hat{x} \in \widehat{E}$ we will use notation $\hat{x} = (x_\kappa)_{\kappa \in H}$ where $x_\kappa \in E_\kappa$, $\kappa \in H$.

(i) If $\hat{a}, \hat{b}, \hat{c}, \hat{d} \in \widehat{E}$ are such that $[\hat{a}, \hat{b}] \cap [\hat{c}, \hat{d}] = \emptyset$ then there exists $\kappa_0 \in H$ such that $[a_{\kappa_0}, b_{\kappa_0}] \cap [c_{\kappa_0}, d_{\kappa_0}] = \emptyset$. Conversely, if for some $\kappa_0 \in H$ and for $a_{\kappa_0}, b_{\kappa_0}, c_{\kappa_0}, d_{\kappa_0} \in E_{\kappa_0}$. we have $[a_{\kappa_0}, b_{\kappa_0}] \cap [c_{\kappa_0}, d_{\kappa_0}] = \emptyset$ then, for all $\kappa \in H$, $\kappa \neq \kappa_0$, we put $a_\kappa = c_\kappa = 0$ and $b_\kappa = d_\kappa = 1$ which gives $[\hat{a}, \hat{b}] \cap [\hat{c}, \hat{d}] = \emptyset$. Thus we conclude that $\hat{E}$ has separated intervals iff all E_κ have separated intervals.

(ii) This follows by (i) and [17, Lemma 2.2] □

Theorem 16. *Every complete atomic distributive effect algebra E has separated intervals and it is compact* (o)*-topological with* $\tau_o = \tau_i$.

Proof. By [22, Theorem 3.1 and Corollary 3.2] we have $E \cong \prod\{[0, p_\kappa] \mid \kappa \in H\}$ where $\{p_\kappa \mid \kappa \in H\}$ is the set of all atoms of $C(E)$ and for every $\kappa \in H$ the interval $[0, p_\kappa]$ is either a finite chain or a distributive diamond $\{0_\kappa, a_\kappa, b_\kappa, p_\kappa\}$ in which $p_\kappa = 2a_\kappa = 2b_\kappa$, hence all $[0, p_\kappa]$ have separated intervals. Moreover, E is (o)-continuous. By Theorem 8, it follows that E is (o)-topological since, in view of Lemma 15, $\tau_o = \tau_i$ is compact Hausdorff. □

Corollary 17. *Every complete atomic MV-effect algebra (MV-algebra) E has separated intervals and it is compact* (o)*-topological with* $\tau_o = \tau_i$.

Note that statements of Theorem 16 and Corollary 17 need not be true for non-atomic E, as no non-atomic complete Boolean algebra is (o)-topological. On the other hand there are non-atomic MV-effect algebras which are compact (o)-topological with $\tau_o = \tau_i$ (see Example 11).

The set A_E of all atoms of an atomic effect algebra E is called *almost orthogonal* if for every $p \in A_E$ the set $\{a \in A_E \mid a \not\leq p'\}$ is finite.

Theorem 18. *Let E be a compact* (o)*-topological atomic effect algebra and A_E is the set of all atoms of E. Then:*

(i) A_E *is almost orthogonal.*

(ii) *For every $p \in A_E$ there are finite elements $u_1, u_2, \dots, u_n$ of E such that* $E = \left(\bigcup_{k=1}^n [u_k, 1]\right) \cup [0, p']$ *and* $[0, p'] \cap \left(\bigcup_{k=1}^n [u_k, 1]\right) = \emptyset$.

Proof. Set $A_E = \{p \in E \mid p \text{ is an atom of } E\}$. Let $p \in A_E$ and $x \in E$, $x \neq 0$. We have proved in [25, Theorem 3.1] that $x = \bigvee\{u \in E \mid u \leq x,\ u \text{ is finite}\}$. It follows that either $x \leq p'$, or there is a finite element $u \in E$ with $u \leq x$ and $u \not\leq p'$. Let $\mathcal{U} = \{u \in E \mid u \text{ is finite}\}$. Then $E = \left(\bigcup_{u \in \mathcal{U}, u \not\leq p'} [u, 1]\right) \cup [0, p']$. Because E is (o)-topological, the intervals $[u, 1]$ and $[0, p']$ are τ_o-clopen, by Theorem 9. As E is compact there is a finite set $\{u_1, u_2, \dots, u_n\} \subseteq \mathcal{U}$, $u_k \not\leq p'$ such that $E = \left(\bigcup_{k=1}^n [u_k, 1]\right) \cup [0, p']$. Evidently $\left(\bigcup_{k=1}^n [u_k, 1]\right) \cap [0, p'] = \emptyset$, as $u_k \not\leq p'$, for $k = 1, \ldots, n$. Moreover, if $a \in A_E$ and $a \not\leq p'$ then there is $k \in \{1, \ldots, n\}$ such that $a = u_k$. It follows that the set $\{a \in A_E \mid a \not\leq p'\}$ is finite. □

Proposition 19. *There are nonatomic complete MV-effect algebras which are compact* (o)*-topological with* $\tau_o = \tau_i$ *(see Example* 11*).*

The next example shows that *a complete atomic effect algebra* E *with compact Hausdorff order topology and* $\tau_i = \tau_o$ *need not be (compact)* (o)*-topological* since E need not be (o)-continuous.

Example 20. *Let* E *be a horizontal sum of MV-effect algebras* M_1 *and* M_2*, which means that we identify least and greatest elements of* M_1 *and* M_2*, respectively, and all pairs* $a \in M_1 \setminus \{0,1\}$ *and* $b \in M_2 \setminus \{0,1\}$ *are noncomparable. Further, let* M_1 *be an MV-effect algebra derived from a Boolean algebra with infinitely many atoms and* $M_2 = \{0, a, 2a = 1\}$*.*

To show that E *is not* (o)*-continuous, we put* $A = \{p \in M_1 \mid p$ *is an atom of* $M_1\}$ *and* $u_\alpha = \bigvee \alpha$*, for all finite sets* $\alpha \subseteq A$*. Then* $u_\alpha \uparrow 1$*, but* $u_\alpha \wedge a = 0$ *while* $1 \wedge a = a$*. It follows that* E *is not* (o)*-topological. Clearly* E *has separated intervals which gives that* $\tau_o = \tau_i$ *is Hausdorff compact topology, as* E *is complete.*

Finally, the next example shows that *an* (o)*-topological complete atomic modular effect algebra* E *need not be compact* (o)*-topological*.

Example 21. *Let the effect algebra* E *be a horizontal sum (0–1 pasting) of contably many distributive diamonds* $E_\kappa = \{0_\kappa, a_\kappa, b_\kappa, 1_\kappa = 2a_\kappa = 2b_\kappa\}$ $\kappa \in H$ *and* H *be infinite. Evidently,* τ_o *is discrete and hence it is not compact since* H *is infinite.*

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Department of Mathematics, Faculty of Electrical Engineering and Information Technology STU, SK-812 19 Bratislava 1

E-mail address: `zriecan@elf.stuba.sk`