On a reliable solution of a Volterra integral equation in a Hilbert Space

Igor Bock, Ján Lovíšek, Bratislava

Summary. We consider a class of Volterra-type integral equation in a Hilbert space. The operators of the problem appear as time-dependent functions with values in the space of linear continuous operators mapping a Hilbert space into its dual. We are looking for maximal values of cost functionals with respect to the admissible set of operators. The existence of a solution in the continuous and the discretized form is verified. The convergence analysis is performed. The results are applied to a quasistationary problem for an anisotropic viscoelastic body made of a long memory material.

Key words: Volterra integral equation in a Hilbert space, Rothe's method, maximization problem, viscoelastic body

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INTRODUCTION

We shall deal with the maximum optimization problem connected with a Volterra integral equation in the Hilbert space. We consider a class of operator-functions $t \to A(t)$ appearing in the state integral equation as the admissible set of control parameters. We shall use the approach similar to [5], where the maximization problem for the class of coefficients of parabolic problems was considered.

In contrast to [5] we start with the abstract formulation of the problem and its approximation in Section 1. We shall verify the existence and uniqueness theorem for certain class of linear continuous operators acting from the Hilbert space into its dual.Applying the Rothe's method (see e.g. [6], [7],

[9]) we state the convergence result for the approximated state problem with respect to a time variable and to a sequence of finite-dimensional subspaces modelling the finite element spaces. In Section 2 we state the maximization problem representing so called "worst scenario" i.e. the worst admissible operators.

The problem formulated in a Hilbert space will be applied to the reliable solution problem for the anisotropic viscoelastic body made of a long memory material. A suitable functional depending on the time and space dependent coefficients is to be maximized. The approximate solution using three dimensional finite elements and the Hermitian interpolation with respect to the time variable is explained.

1 The state problem and its approximation

For any Banach space X and $T > 0$ we introduce the set $L^\infty(0,T;X)$ of all measurable essentially bounded functions $w : [0,T] \to X$, $C([0,T],X)$ the set all continuous functions and the Sobolev space

$$W^{1,\infty}(0,T;X) = \{w \in L^\infty(0,T;X) : w' \in L^\infty(0,T;X)\}$$

with a derivative w' in the sense of distribitions. All sets of functions are Banach spaces with norms

$$\|w\|_{L^\infty(0,T;X)} = \operatorname{ess}\sup_{t\in[0,T]} \|w(t)\|_X,\ \|w\|_{C([0,T],X)} = \max_{t\in[0,T]} \|w(t)\|_X,$$
$$\|w\|_{W^{1,\infty}(0,T;X)} = \|w\|_{L^\infty(0,T;X)} + \|w'\|_{L^\infty(0,T;X)}.$$

We have the continuous imbedding $W^{1,\infty}(0,T;X) \subset C([0,T],X)$. Every element $w \in W^{1,\infty}(0,T;X)$ can be expressed in a form

$$w(t) = w(0) + \int_0^t w'(s)ds,\ t \in [0,T].$$

Let V be a Hilbert space with a scalar product $((.,))$ and a norm $\|.\|$, V^* its dual space with a norm $\|.\|_*$. We denote by $\langle f, v\rangle$ the duality pairing between the functional $f \in V^*$ and the element $v \in V$.

We shall deal with the set of operator functions $t \to A(t)$ with values in the Banach space $\mathcal{B} = \mathcal{L}(V,V^*)$ of all linear bounded operators $A : V \to V^*$.

We assume moreover that $A \in \mathcal{U}$, where $\mathcal{U} = W^{1,\infty}(0,T;\mathcal{B})$. The operator $A(0) : V \to V^*$ is assumed to be positively definite i.e.

$$\langle A(0)v, v\rangle \geq \alpha_0 \|v\|^2 \ \forall v \in V, \ \alpha_0 > 0. \tag{1}$$

We introduce a norm in $\mathcal{U}$ equivalent with the original norm in $W^{1,\infty}(0,T;\mathcal{B})$ by

$$\|A\|_{\mathcal{U}} = \|A(0)\|_{\mathcal{B}} + \operatorname*{ess\,sup}_{t\in[0,T]} \|A'(t)\|_{\mathcal{B}}.$$

Let $f : [0,T] \to V^*$, $(A' * u)(t) = \int_0^t A'_t(t-s)u(s)ds$. We consider

The state problem:

To find $u : [0,T] \to V$ fulfilling

$$A(0)u(t) + (A' * u)(t) = f(t), \ t \in [0,T]. \tag{2}$$

Theorem 1.1 *Let $f \in C([0,T],V^*)$. Then there exists a unique solution $u \in C([0,T],V)$ of the equation (2).*

Proof. There exists due to Lax-Milgram theorem the inverse operator $A(0)^{-1} \in \mathcal{L}(V^*,V)$. The equation (2) is then equivalent to the Volterra integral equation in a Banach space V:

$$u(t) + (B * u)(t) = q(t), \ t \in [0,T], \tag{3}$$

with $B \in L^\infty(0,T;\mathcal{L}(V,V))$, $q \in C([0,T],V)$ defined by $B(t) = A(0)^{-1}A'(t)$, $q(t) = A(0)^{-1}f(t)$, $t \in [0,T]$.

The equation (3) can be expressed in a form

$$u = \mathcal{A}(u), \tag{4}$$

where $\mathcal{A} : C([0,T],V) \to C([0,T],V)$ is defined by $\mathcal{A}(u) = q - B * u$.

It can seen easily that there exists the integer $n \equiv n(B,T)$ such that the operator $\mathcal{A}^n$ is contractive in the Banach space $C([0,T],V)$. More precisely, there holds

$$\|\mathcal{A}^n u - \mathcal{A}^n v\| \leq \frac{T^n \|B\|^n_{L^\infty(0,T;\mathcal{L}(V,V))}}{n!}, \ n = 1,2,...$$

and hence there exist $n_0 \in N$ and $\kappa \in (0,1)$ such that

$$\|\mathcal{A}^n u - \mathcal{A}^n v\| \leq \kappa \|u - v\| \ \forall u,v \in V, \ n \geq n_0.$$

Applying the Banach fixed point theorem we obtain the existence and uniqueness of a solution of (4) which is also a unique solution $u \in C([0,T],V)$ of (3) and (2).

We shall continue with a full discretization of the problem (2). Let us assume a family of finite-dimensional subspaces $\{V_h\},\ V_h \subset V,\ h \in (0,h_0)$ such that for any $v \in V$ there exist $v_h \in V_h,\ h \in (0,h_0)$ fulfilling

$$v_h \to v \text{ in } V \text{ as } h \to 0+\,. \tag{5}$$

Let $A^h \in W^{1,\infty}(0,T;\mathcal{B}),\ h \in (0,h_0)$ be approximating operators satisfying

$$\langle A^h(0)u,u\rangle \geq \alpha_0\|u\|^2\ \forall u \in V \text{ with } \alpha_0 > 0, \tag{6}$$

$$A^h \to A \text{ in } \mathcal{U} \text{ as } h \to 0+\,. \tag{7}$$

Further we assume for $\tau \in (0,\tau_0)$ the division of the interval $[0,T]$ by

$$0 = t_0 < t_1 < ... < t_{N-1} < t_N = T,\ t_i = i\tau,\ i = 1,...,N \equiv N(\tau).$$

We define the approximation $u^{h\tau} \in C([0,T],V_h)$ of a solution u of (2) by

$$u^{h\tau}(t) = u^{h\tau}_{i-1} + \frac{t-t_{i-1}}{\tau}(u^{h\tau}_i - u^{h\tau}_{i-1}),\ t \in [t_{i-1},t_i], \tag{8}$$

where $\{u^{h\tau}_i\}$ are unique solutions (due to Lax-Milgram theorem) of stationary problems

$$\langle A^h_0 u^{h\tau}_0, v\rangle = \langle f^h_0, v\rangle\ \forall v \in V_h, \tag{9}$$

$$\langle A^h_0 u^{h\tau}_i + \sum_{j=0}^{i-1}(A^h_{i-j} - A^h_{i-j-1})u^{h\tau}_j, v\rangle = \langle f^h_i, v\rangle\ \forall v \in V_h, \tag{10}$$

$$i = 1,...,N.$$

with $A^h_i = A^h(t_i),\ f^h_i = f^h(t_i),\ i = 0,1,...,N$. We introduced $f^h : [0,T] \to V^*$ - approximating functionals of f.

In order to assure the convergence of the scheme we impose the smoothness condition on the right-hand side f.

Theorem 1.2 *Let $f \in W^{1,\infty}([0,T],V^*),\ f^h \in W^{1,\infty}([0,T],V^*),\ h \in (0,h_0)$ be such that*

$$f^h \to f \text{ in } W^{1,\infty}([0,T],V^*) \text{ as } h \to 0+\,. \tag{11}$$

Then

$$u^{h\tau} \rightharpoonup^* u \text{ in } W^{1,\infty}([0,T],V) \text{ as } h \to 0+,\ \tau \to 0+, \tag{12}$$

where $u \in W^{1,\infty}(0,T;V)$ *is a unique solution of the equation (2) and* $u^{h\tau}$ *is defined by (8)-(10).*

If a solution u fulfils the condition

$$\pi_h(u) \to u \text{ in } C([0,T],V) \text{ as } h \to 0+, \tag{13}$$

where $\pi_h(u)(t) \in V_h,\ t \in [0,T]$ *is the orthogonal projection of* $u(t)$ *onto the subspace* V_h*, then*

$$u^{h\tau} \to u \text{ in } C([0,T],V) \text{ as } h \to 0+,\ \tau \to 0+\,. \tag{14}$$

Proof. Using the uniform coercivity (6) and the convergence (7) we obtain from (10) the inequalities

$$\alpha_0 \|u_i^{h\tau}\|^2 \leq \langle \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (A^h)'_s(t_i - s)ds\ u_j^{h\tau} + f_i^h, u_i^{h\tau} \rangle$$

and

$$\|u_i^{h\tau}\| \leq \alpha_0^{-1}(\|A\|_{\mathcal{U}} + \epsilon) \sum_{j=0}^{i-1} \tau \|u_j^{h\tau}\| + \alpha_0^{-1}\|f_i^h\|_*,\ i = 1, ..., N,\ h \in (0, h_0).$$

Applying the convergence (7), (11) and a discrete form of Gronwall's lemma ([6]) with respect to $\{u_i^{h\tau}\}$ we obtain the a priori estimate

$$\|u_i^{h\tau}\| \leq C_1(T),\ i = 0, 1, ..., N(\tau),\ h \in (0, h_0),\ \tau \in (0, \tau_0). \tag{15}$$

Let us denote $\delta w_i = \frac{1}{\tau}(w_i - w_{i-1}),\ w_i = w(t_i)$ for any function $w \in C([0,T],X)$ with values in a Banach space X. Setting $i-1$ instead of i into (10) and substracting from (10) we obtain the relations

$$\langle A_0^h \delta u_i^{h\tau}, v \rangle = \langle -\tau \sum_{j=0}^{i-2} \delta A_{i-j-1}^h \delta u_j^{h\tau}\ - \delta A_i^h u_0^{h\tau} + \delta f_i^h, v \rangle =$$

$$\langle \sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} (A^h)'_s(t_{i-1} - s)ds \delta u_j^{h\tau} - \frac{1}{\tau} \int_{t_{i-1}}^{t_i} (A^h)'(s)ds\ u_0^{h\tau} + \delta f_i^h, v \rangle\ \forall v \in V_h.$$

Again using the convergence (7), (11) and a discrete form of Gronwall's lemma we arrive at the estimate

$$\|\delta u_i^{h\tau}\| \le C_2(T), \ i = 0, 1, ..., N(\tau), \ h \in (0, h_0), \ \tau \in (0, \tau_0). \tag{16}$$

Let us define the step functions $\bar{u}^{h\tau}, \ \tilde{u}^{h\tau} : [0, T] \to V$ by

$$\begin{aligned}
&\bar{u}^{h\tau}(0) = u_0^{h\tau}, \ \bar{u}^{h\tau}(t) = u_i^{h\tau}, \quad t \in (t_{i-1}, t_i], \\
&\tilde{u}^{h\tau}(0) = 0, \ \tilde{u}^{h\tau}(t) = u_{i-1}^{h\tau}, \quad t \in (t_{i-1}, t_i], \ i = 1, ..., N.
\end{aligned}$$

The equation (10) can be expressed in a form

$$\begin{aligned}
&\langle A^h(0)\bar{u}^{h\tau}(t) + (A^h)'_t * \tilde{u}^{h\tau}(t) + \int_t^{t_i} (A^h)'_t(t_i - s)\tilde{u}^{h\tau}(s)ds, v\rangle \\
&= \langle \int_0^t [(A^h)'_t(t - s) - (A^h)'_t(t_i - s)]\tilde{u}^{h\tau}(s)ds + \bar{f}^{h\tau}(t), v\rangle \\
&\quad \text{for all } v \in V_h, \ t \in (t_{i-1}, t_i], \ i = 1, ..., N.
\end{aligned} \tag{17}$$

The a priori estimates (15), (16) imply the estimate

$$\|u^{h\tau}\|_{W^{1,\infty}(0,T;V)} \le C_3(T), \ h \in (0, h_0), \ \tau \in (0, \tau_0) \tag{18}$$

and the existence of a function $w \in W^{1,\infty}(0, T; V)$ and a sequence $\{h_n, \tau_n\}$, $h_n > 0, \ \tau_n > 0$ fulfilling

$$h_n \to 0, \ \tau_n \to 0, \quad u^{h_n\tau_n} \rightharpoonup^* w \text{ in } W^{1,\infty}(0, T; V). \tag{19}$$

Simultaneously there hold the inequalities

$$\|u^{h\tau}(t) - \bar{u}^{h\tau}(t)\| \le \tau C_2(T), \tag{20}$$

$$\|u^{h\tau}(t) - \tilde{u}^{h\tau}(t)\| \le \tau C_2(T). \tag{21}$$

Applying the assumptions (5), (7), (11), the a priori estimate (15) and the estimates (20), (21) we obtain from the equation (17) that a limiting function w from (19) is a solution of the state equation (2). We have $w \equiv u$ due to the uniqueness of a solution and hence the convergence (12) holds. We remark that we have used the relation

$$\lim_{\tau \to 0} \|F(t + \tau) - F(t)\|_{L^p(0,T;X)} = 0, \ 1 \le p < \infty$$

for any function $F \in L^p(0, T; X)$ extended by $F(t + \tau) = 0$, if $t + \tau \notin [0, T]$ in performing the limit in the integral on right-hand side of the relation (17).

It remains us to prove the uniform convergence (14). Let $u_h(t) := \pi_h u(t) \in V_h,\ t \in [0,T]$ be the orthogonal projection onto V_h. It fulfils due to the assumption (13) the uniform convergence

$$\lim_{h\to 0+} \|u - u_h\|_{C([0,T],V)} = 0. \tag{22}$$

We define functions $v_{h\tau} : [0,T] \to V_h$ by

$$v_{h\tau} = u_h - \tilde{u}^{h\tau},\ h \in (0,h_0),\ \tau \in (0,\tau_0). \tag{23}$$

Taking into account the relations (2), (17) we obtain the identity

$$\langle A^h(0)v_{h\tau}(t) + (A^h)' * v_{h\tau}(t), v_{h\tau}(t)\rangle = \langle \omega_{h\tau}(t), v_{h\tau}(t)\rangle,\ t \in [0,T], \tag{24}$$

where

$$\begin{aligned}
&\omega_{h\tau}(t) = A^h(0)u_h(t) - A(0)u(t) + (A^h)' * u_h(t) - A' * u(t)\\
&- \int_t^{t_i} (A^h)'_s(t_i - s)\tilde{u}^{h\tau}(s)ds + \int_0^t [(A^h)'_s(t-s) - (A^h)'_s(t_i - s)]\tilde{u}^{h\tau}(s)ds\\
&+ A^h(0)[\tilde{u}_h(t) - \bar{u}_h(t)] + f(t) - \bar{f}^{h\tau}(t).
\end{aligned}$$

The uniform coercivity (6) and the assumption $A' \in L^\infty(0,T;\mathcal{B})$ imply the inequality

$$\|v_{h\tau}(t)\| \le \|\omega_{h\tau}(t)\|_* + C_3(T)\int_0^t \|v_{h\tau}(s)\|ds\ \forall t \in [0,T].$$

The estimate

$$\|v_{h\tau}(t)\| \le \|\omega_{h\tau}(t)\|_* \exp TC_3(T)\ \forall t \in [0,T] \tag{25}$$

follows due to Gronwall's lemma. The previous assumptions and estimates imply the convergence

$$\omega_{h\tau} \to 0 \text{ in } L^\infty(0,T;V^*) \text{ as } h \to 0+,\ \tau \to 0+.$$

The uniform convergence (14) then follows from (22), (23), (25) and the proof is complete.

2 A maximization problem and its approximation

Let us ussume the compact subset $\mathcal{U}_{ad} \subset \mathcal{U}$ of operator functions $A : [0, T] \to \mathcal{B}$ such that $A(0)$ fulfil uniform positive definiteness (1). The functional $\Phi : \mathcal{U} \times C([0, T]; V) \to R$ fulfils

$$A_n \in \mathcal{U}_{ad},\ \{A_n, u_n\} \to \{A, u\} \text{ in } \mathcal{U} \times C([0, T]; V) \text{ as } n \to \infty$$
$$\Longrightarrow \lim_{n\to\infty} \sup \Phi(A_n, u_n) \leq \Phi(A, u). \tag{26}$$

We formulate

Maximization Problem:

$$A_* = \arg \max_{A\in\mathcal{U}_{ad}} \Phi(A, u(A)), \tag{27}$$

where $u(A)$ is a solution of the integral equation (2).

Theorem 2.1 *Let the assumptions of Theorem 1.1 be fulfilled. Let the functional Φ satisfy (26).*

Then the Maximization Problem (27) has at least one solution.

Proof. Let $\{A_n\} \subset \mathcal{U}_{ad}$ be a maximizing sequence for the problem (27) i.e.,

$$\lim_{n\to\infty} \Phi(A_n, u(A_n)) = \sup_{A\in\mathcal{U}_{ad}} \Phi(A, u(A)). \tag{28}$$

There exists its subsequence (again denoted by $\{A_n\}$) and $A_* \in \mathcal{U}_{ad}$ such that

$$A_n \to A_* \text{ in } \mathcal{U}. \tag{29}$$

The corresponding sequence $\{u_n\}$ fulfils the equations

$$A_n(0)u_n + A'_n * u_n = f,\ n = 1, 2, ... \tag{30}$$

Let us denote by $u_* \in C([0, T], V) \equiv u(A_*)$ a unique solution of the equation

$$A_*(0)u_* + A'_* * u_* = f. \tag{31}$$

If

$$u_n \to u_* \text{ in } C([0, T], V), \tag{32}$$

then the property (26) implies the relation (27).

Comparing (30) and (31) we arrive at the equation

$$A_n(0)(u_n - u_*)(t) + \int_0^t (A_n')_t(t-s)(u_n - u_*)(s)ds = \omega_n(t), \; t \in [0,T] \quad (33)$$

with

$$\omega_n(t) = [A_*(0) - A_n(0)]u_*(t) + \int_0^t (A_* - A_n)_t'(t-s)u_*(s)ds.$$

We have

$$\lim_{n\to\infty} \|\omega_n\|_{C([0,T],V^*)} = 0 \quad (34)$$

due to the convergence (29). The equation (33) implies due to the uniform coercivity and boundedness of $\{A_n\}$ in $\mathcal{U}$ the inequality

$$\|(u_n - u_*)(t)\| \le M \int_0^t \|(u_n - u_*)(s)\|ds + \|\omega_n(t)\|_* \; \forall t \in [0,T].$$

Applying the Gronwall lemma we arrive at the estimate

$$\|(u_n - u_*)(t)\| \le C_4(M,T)\|\omega_n(t)\|_* \; \forall t \in [0,T]$$

and the uniform convergence (32) follows due to (34). The convergence (28), (29), (32) together with the property (26) implies that a function $A_* \in \mathcal{U}_{ad}$ solves the Maximization problem (27).

We continue with an approximate maximization problem. We assume that there hold the assumptions of Theorem 1.2.

Let $\mathcal{U}_{ad}^h \subset \mathcal{U}_{ad}, \; h \in (0,h_0)$ be such compact subsets that for all $A \in \mathcal{U}_{ad}$ and $h \in (0,h_0)$ there exist approximating operator functions $A_h \in \mathcal{U}_{ad}^h$ fulfilling the convergence (7). Let $\tau \in (0,\tau_0)$. We assume that the functional $\Phi : \mathcal{U} \times C([0,T],V) \to R$ fulfils the continuity property

$$\begin{aligned} &A_k \in \mathcal{U}_{ad}, \; u_k \in V, \; \{A_k, u_k\} \to \{A, u\} \text{ in } \mathcal{U} \times C([0,T],V) \text{ as } k \to \infty \\ &\Longrightarrow \lim_{k\to\infty} \Phi(A_k, u_k) = \Phi(A,u). \end{aligned} \quad (35)$$

For $A \in \mathcal{U}_{ad}^h$ we set $u^{h\tau}(A) \in W^{1,\infty}(0,T;V)$ a solution belonging to the approximating problem (9), (10).

The Approximate Maximization Problem $\mathbf{P}^h$:

$$A_*^{h\tau} = \arg\max_{A\in\mathcal{U}_{ad}^h} \Phi(A, u^{h\tau}(A)). \quad (36)$$

Theorem 2.2 *Let $f \in W^{1,\infty}(0,T;V^*)$ and the admissible sets $\mathcal{U}_{ad}$, $\mathcal{U}_{ad}^h$ satisfy the assumptions stated above. Let the assumption (13) be fulfilled for every $A \in \mathcal{U}$. Then there exists a solution $A_*^{h\tau} \in \mathcal{U}_{ad}^h$ of the Problem (36).*

If A_* is a solution of the Problem (27) and a sequence $\{h_n, \tau_n\}$ is such that,

$$h_n > 0,\ \tau_n > 0,\ h_n \to 0,\ \tau_n \to 0,$$

then there exists its subsequence $\{h_k, \tau_k\}$ fulfilling

$$A_*^{h_k\tau_k} \rightharpoonup^* A_* \text{ in } \mathcal{U} \text{ for } k \to \infty. \tag{37}$$

Proof. Let $\{A_n\} \subset \mathcal{U}_{ad}^h$ be a maximizing sequence for the problem (36) i.e.

$$\lim_{n\to\infty} \Phi(A_n, u^{h\tau}(A_n)) = \sup_{A\in\mathcal{U}_{ad}^h} \Phi(A, u^{h\tau}(A)). \tag{38}$$

There exists its subsequence (again denoted by $\{A_n\}$) and $A_*^{h\tau} \in \mathcal{U}_{ad}^h$ such that

$$A_n \to A_*^{h\tau} \text{ in } \mathcal{U}. \tag{39}$$

The corresponding sequence $\{u_n^{h\tau}\},\ u_n^{h\tau} = u^{h\tau}(A_n)$ fulfils the relation analogous to (17)

$$\begin{aligned}
&\langle A_n(0)\bar{u}_n^{h\tau}(t) + A_n' * \tilde{u}_n^{h\tau}(t) + \int_t^{t_i} (A_n)_t'(t_i - s)\tilde{u}_n^{h\tau}(s)ds, v\rangle \\
&= \langle \int_0^t [(A_n)_t'(t-s) - (A_n)_t'(t_i - s)]\tilde{u}_n^{h\tau}(s)ds + \bar{f}^{h\tau}(t), v\rangle \\
&\quad \text{for all } v \in V_h,\ t \in (t_{i-1}, t_i],\ i = 1, ..., N.
\end{aligned} \tag{40}$$

Let $u_*^{h\tau}$ be a solution of the approximated scheme corresponding to $A_*^{h\tau}$:

$$\begin{aligned}
&\langle A_*^{h\tau}(0)\bar{u}_*^{h\tau}(t) + (A_*^{h\tau})' * \tilde{u}_*^{h\tau}(t) + \int_t^{t_i} (A_*^{h\tau})_t'(t_i - s)\tilde{u}_*^{h\tau}(s)ds, v\rangle \\
&= \langle \int_0^t [(A_*)_t'(t-s) - (A_*^{h\tau})_s'(t_i - s)]\tilde{u}_*^{h\tau}(s)ds + \bar{f}^{h\tau}(t), v\rangle \\
&\quad \text{for all } v \in V_h,\ t \in (t_{i-1}, t_i],\ i = 1, ..., N.
\end{aligned} \tag{41}$$

The following estimates can be verified in the same way as in the proof of Theorem 1.2:

$$\|u_n^{h\tau}\|_{W^{1,\infty}(0,T;V)} \le C_5(T),$$

$$\begin{aligned}
&\|u_n^{h\tau}(t)-\bar{u}_n^{h\tau}(t)\| \le \tau C_6(T),\\
&\|u_n^{h\tau}(t)-\tilde{u}_n^{h\tau}(t)\| \le \tau C_6(T),\\
&\|u_*^{h\tau}\|_{W^{1,\infty}(0,T;V)} \le C_5(T),\\
&\|u_*^{h\tau}(t)-\bar{u}_*^{h\tau}(t)\| \le \tau C_6(T),\\
&\|u_*^{h\tau}(t)-\tilde{u}_*^{h\tau}(t)\| \le \tau C_6(T),\ n>n_0,\ h\in(0,h_0),\ \tau\in(0,\tau_0).
\end{aligned} \tag{42}$$

Let us denote

$$v_n^{h\tau} = \tilde{u}_n^{h\tau} - u_*^{h\tau},\ n>n_0,\ h\in(0,h_0),\ \tau\in(0,\tau_0). \tag{43}$$

We obtain from (40), (41) the identity

$$\langle A_n(0)v_n^{h\tau}(t) + A_n' * v_n^{h\tau}(t), v_n^{h\tau}(t)\rangle = \langle \omega_n^{h\tau}(t), v_n^{h\tau}(t)\rangle. \tag{44}$$

with $\omega_n^{h\tau} \in C([0,T],V^*)$ fulfilling

$$\lim_{n\to\infty} \|\omega_n^{h\tau}\|_{C([0,T],V^*)} = 0. \tag{45}$$

Applying the uniform coercivity of the operators $\{A_n\}$ and the Gronwall's lemma in (44) we obtain due to (42), (43), (45) the convergence

$$u_n^{h\tau} \to u_*^{h\tau} \text{ in } C([0,T],V). \tag{46}$$

The property (26) of the functional Φ and the convergence (38), (39), (46) then imply that $A_*^{h\tau}$ is a solution of the Approximate Maximization Problem (36).

We continue with the convergence of the method. Let $h_n > 0, \tau_n > 0,\ h_n \to 0,\ \tau_n \to 0$. The sequence $\{A_*^{h_n\tau_n}\}$ belongs to the compact set $\mathcal{U}_{ad} \subset \mathcal{U}$. Then there exist its subsequence $\{A_*^{h_k\tau_k}\}$ and the operator function $A_0 \in \mathcal{U}_{ad}$ fulfilling

$$A_*^{h_k\tau_k} = \arg\max_{A\in\mathcal{U}_{ad}^{h_k}} \Phi(A, u^{h_k\tau_k}(A)), \tag{47}$$

$$A_*^{h_k\tau_k} \to A_0 \text{ in } \mathcal{U}. \tag{48}$$

Let $u_0 \equiv u(A_0)$ be a unique solution of the state equation

$$A_0 u_0 + A_0 * u_0 = f \tag{49}$$

and $u_*^{h_k\tau_k} \equiv u^{h_k\tau_k}(A_*^{h_k\tau_k}),\ k = 1, 2...$ be a unique solution of the approximate problem

$$\langle A_*^{h_k\tau_k}(0)\bar{u}_*^{h_k\tau_k}(t) + (A_*^{h_k\tau_k})'_t * \tilde{u}_*^{h_k\tau_k}(t) + \int_t^{t_i}(A_*^{h_k\tau_k})'_t(t_i - s)\tilde{u}_*^{h_k\tau_k}(s)ds, v\rangle$$
$$= \langle \int_0^t [(A_*^{h_k\tau_k})'_t(t-s) - (A_*^{h_k\tau_k})'_t(t_i - s)]\tilde{u}_*^{h_k\tau_k}(s)ds + \bar{f}^{h_k\tau_k}(t), v\rangle$$
$$\text{for all } v \in V_{h_k},\ t \in (t_{i-1}^k, t_i^k],\ t_i^k = i\tau_k,\ i = 1, ..., N_k.$$

Using the same approach as in the proof of Theorem 1.2 we obtain the convergence

$$u_*^{h_k\tau_k}(A_*^{h_k\tau_k}) \to u_0 \equiv u(A_0) \text{ in } C([0,T], V) \text{ as } k \to \infty. \tag{50}$$

For an arbitrary $A \in \mathcal{U}_{ad}$ there exists a sequence $\{\tilde{A}_k\} \in \mathcal{U}_{ad}^{h_k}$ fulfilling $\tilde{A}_k \to A$ in $\mathcal{U}$. Simultaneously we have

$$u^{h_k\tau_k}(\tilde{A}^k) \to u(A) \text{ in } C([0,T], V) \text{ as } k \to \infty.$$

The relations (47)-(50) and the continuity property (35) then imply the relations

$$\Phi(A_0, u(A_0)) \ge \lim_{k\to\infty} \sup \Phi(A_*^{h_k\tau_k}, u_*(A_*^{h_k\tau_k})$$
$$\ge \lim_{k\to\infty} \Phi(\tilde{A}_k, u^{h_k\tau_k}(\tilde{A}_k)) = \Phi(A, u(A)).$$

Then we obtain $A_0 \equiv A_*$ is a solution of the Maximization Problem (27). Simultaneously there holds the convergence (37) and the proof is complete.

3 Applications to Maximization Problems for Viscoelastic Bodies

Let $\Omega \subset R^3$ be a a bounded domain with a Lipschitz boundary $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$ with open in $\partial\Omega$ parts Γ_0, Γ_1, $\text{meas}(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and the unit outward normal vector $\mathbf{n}(\mathbf{x})$, $\mathbf{x} \in \partial\Omega$. We assume a quasistationary state of a viscoelastic body occupying Ω and acting upon body forces $\mathbf{f}(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, $t \in [0, T]$ and surface tractions $\mathbf{g}(\mathbf{x}, t)$, $\mathbf{x} \in \Gamma_1$, $t \in [0, T]$. Considering the Boltzman type anisotropic long memory material ([3]) we obtain the equilibrium equations

$$-\text{div }\sigma(\mathbf{u}; \mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t),\ \mathbf{x} \in \Omega,\ t \in [0, T] \tag{51}$$

with boundary conditions

$$\mathbf{u}(\mathbf{x},t)=\mathbf{0},\ \mathbf{x}\in\Gamma_0,\quad \sigma(\mathbf{u};\mathbf{x},t)\mathbf{n}=\mathbf{g}(\mathbf{x},t),\ \mathbf{x}\in\Gamma_1, \tag{52}$$

and stress-strain relations

$$\sigma_{ij}(\mathbf{u};\mathbf{x},t)=$$
$$A_{ijkl}(\mathbf{x},0)\epsilon_{kl}(\mathbf{u}(t))+\int_0^t\frac{\partial}{\partial t}A_{ijkl}(\mathbf{x},t-s)\epsilon_{kl}(\mathbf{u}(s))ds, \tag{53}$$
$$\epsilon_{ij}(\mathbf{u})=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\ i,j=1,2,3. \tag{54}$$

We assume the components of fourth order tensor functions $A_{ijkl}(.,.):\Omega\times[0,T]\to R$ to fulfil

$$A_{ijkl}\in W^{1,\infty}(0,T;L^\infty(\Omega))) \tag{55}$$

The fourth order tensors $A_{ijkl}(\mathbf{x},0)$ are assumed to be uniformly positively definite

$$A_{ijkl}(\mathbf{x},0)\epsilon_{ij}\epsilon_{kl}\geq c_0\epsilon_{ij}\epsilon_{ij},\ c_0>0,\ \text{a.e. in }\Omega,\ \forall\{\epsilon_{ij}\}\in R^{3\times3}_{sym}, \tag{56}$$

where $R^{3\times3}_{sym}$ is the space of all symmetric tensors $\{\epsilon_{ij}\}\in R^{3\times3}$.

After setting

$$V=\{\mathbf{v}\in H^1(\Omega)^3:\ \mathbf{v}(\mathbf{x})=\mathbf{0},\ \mathbf{x}\in\Gamma_0\}$$

the Hilbert space of displacements vectors $\mathbf{v}:\Omega\to R^3$ and using the notations from the previous section we introduce the operator function $A\in\mathcal{U}$ by

$$\langle A(t)\mathbf{u},\mathbf{v}\rangle=\int_\Omega A_{ijkl}(\mathbf{x},t)\epsilon_{ij}(\mathbf{u})\epsilon_{kl}(\mathbf{v})dx,\ \mathbf{u},\ \mathbf{v}\in V. \tag{57}$$

The operator $A(0):V\to V^*$ is positively definite with some constant $\alpha_0>0$ due to the uniform positive-definiteness of the tensor function $\{A_{ijkl}(.,0)\}$ and the Korn's inequality, verified in ([8]). If we define the functional $f(t)\in V^*,\ t\in[0,T]$ by

$$\langle f(t),\mathbf{v}\rangle=\int_\Omega\mathbf{f}(\mathbf{x},t).\mathbf{v}(\mathbf{x})dx+\int_{\Gamma_1}\mathbf{g}(\mathbf{r},t).\mathbf{v}(\mathbf{r})dr,\ v\in V \tag{58}$$

we can identify the state integral equation (2) with a weak formulation of the boundary value problem (51)-(54). Applying Theorem 1.1 we obtain

Theorem 3.1 *Let* $\mathbf{f} \in C([0,T], L^2(\Omega)^3)$, $\mathbf{g} \in C([0,T], L^2(\Gamma_1)^3)$. *Then there exists a unique weak solution* $\mathbf{u} \in C([0,T], V)$ *of the problem (51)-(54).*

Let us introduce the following spaces of coefficients $\{A_{ijkl}\}$ by

$$\mathcal{U} = [W^{1,\infty}(0,T; L^\infty(\Omega))]^{81} \tag{59}$$

and

$$\mathcal{V} = [W^{1,p}(0,T; W^{1,q}(\Omega)) \cap W^{2,p}(0,T; L^\infty(\Omega))]^{81},\ p > 1,\ q > 3 \tag{60}$$

or

$$\mathcal{V} = [W^{1,\infty}(0,T; W^{1,q}(\Omega)) \cap W^{2,p}(0,T; L^1(\Omega))]^{81},\ p > 1. \tag{61}$$

There holds in both cases the compact imbedding $\mathcal{V} \subset\subset \mathcal{U}$.

We have applied the compact imbedding $W^{1,q}(\Omega) \subset\subset L^\infty(\Omega)$ and the theory of compact sets in the spaces $L^p(0,T;B)$, $1 \le p \le \infty$, ($B-$ a Banach space) due to Simon ([11]).

The set of admissible coefficients

$$\begin{aligned} \mathcal{U}_{ad} = \quad & \{\ \{A_{ijkl}\} \in \mathcal{V}:\ \|\{A_{ijkl}\}\|_{\mathcal{V}} \le c_1, \\ & A_{ijkl}(\mathbf{x},0)\epsilon_{ij}\epsilon_{kl} \ge c_0\epsilon_{ij}\epsilon_{ij},\ c_0 > 0,\ \forall \mathbf{x} \in \Omega,\ \forall\{\epsilon_{ij}\} \in R^{3\times 3}_{sym}\} \end{aligned} \tag{62}$$

is compact in the Banach space $\mathcal{U}$.

We can consider instead of the set $\mathcal{U}_{ad}$ its arbitrary convex closed (in $\mathcal{V}$) subset.

Most of viscoelastic materials are described by coefficients fulfilling the exponential decreasing of their time derivatives. In that case we can consider as the admissible set

$$\mathcal{U}^1_{ad} = \{\ \{A_{ijkl}\} \in \mathcal{U}_{ad}:\ \|\{A'_{ijkl}(t)\}\|_{[L^\infty(\Omega)]^{81}} \le c_2 e^{-\beta t},\ \beta > 0,\ \forall t \in [0,T]\}.$$

Very important special case of the set $\mathcal{U}^1_{ad}$ is the set of coefficients in the exponential form

$$A_{ijkl}(t) = B^{\langle 0\rangle}_{ijkl} + \sum_{m=1}^{M} B^{\langle m\rangle}_{ijkl} e^{-\beta_m t},\ \beta_m > 0,\ m = 1,...,M$$

with positively definite fourth-order tensor $\{A_{ijkl}(0)\}$. Precisely, the admissible set has the form

$$\begin{aligned}\mathcal{U}^2_{ad} = \quad & \{\ [\{B^{\langle n\rangle}_{ijkl}\},\{\beta_m\}] \in [W^{1,q}(\Omega)^{81}\times R]^M,\ q>3:\\ & \sum_{n=0}^{M} B^{\langle m\rangle}_{ijkl}(\mathbf{x})\epsilon_{ij}\epsilon_{kl} \geq \alpha_0\epsilon_{ij}\epsilon_{ij},\ \alpha_0>0,\ \forall\mathbf{x}\in\Omega,\ \forall\{\epsilon_{ij}\}\in R^{3\times3}_{sym},\\ & \|\{B^{\langle m\rangle}_{ijkl}\}\|_{W^{1,q}(\Omega)^{81}} \leq c_m,\ n=0,...,M;\\ & 0<\gamma_m\leq\beta_m\leq\delta_m,\ m=1,...,M\}.\end{aligned}$$

Let

$$\bar\Omega = \bigcup_{m=1}^{M}\bar\Omega_m,\ \Omega_i\cap\Omega_j=\emptyset,\ \text{for } i\neq j.$$

We assume the cofficients to be constant with respect to $\mathbf{x}$ on the subsets $\Omega_m,\ m=1,...,M$. The admissible set has then the form

$$\begin{aligned}&\mathcal{U}^3_{ad} = \{\{A_{ijkl}\}\in\mathcal{U} : A_{ijkl}|_{\Omega_m}(\mathbf{x},t) = A^{\langle m\rangle}_{ijkl}(t),\\ &A^{\langle m\rangle}_{ijkl}(0)\epsilon_{ij}\epsilon_{kl}\geq c_0\epsilon_{ij}\epsilon_{ij},\ c_0>0,\ \forall\epsilon_{ij}\in R^{3\times3}_{sym},\\ &A^{\langle m\rangle}_{ijkl}\in W^{2,p}(0,T),\ \|A^{\langle m\rangle}_{ijkl}\|_{W^{2,p}(0,T)}\leq c_m,\ m=1,...,M\}.\end{aligned}$$

We can formulate

Maximization problem $\mathcal{P}$:

$$\mathcal{A}_* = \arg\max_{\mathcal{A}\in\mathcal{U}_{ad}}\Phi(\mathcal{A},\mathbf{u}(\mathcal{A})),\ \mathcal{A}=\{A_{ijkl}\}.$$

with goal functionals $\Phi_i : \mathcal{U}\times C([0,T];V)\rightarrow R,\ i=1,2$ fulfilling the assumptions (26).

Let $\Omega_j\subset\Omega,$ intervals $I_j\subset[0,T],\ j=1,...,J.$

1) $\Phi_1(\mathcal{A},\mathbf{u}(\mathcal{A})) = \max_{1\leq j\leq J}\psi_j(\mathbf{u}(\mathcal{A}))$ with
 a) $\psi_j(\mathbf{u}(\mathcal{A}) = (\text{meas }\Omega_j)^{-1}\int_{\Omega_j}\mathbf{u}(\mathcal{A})(t_*)dx,\ t_*\in(0,T]$, or
 b) $\psi_j(\mathbf{u}(\mathcal{A}) = (\text{meas}I_j)^{-1}(\text{meas}\Omega_j)^{-1}\int_{I_j}\int_{\Omega_j}\mathbf{u}(\mathcal{A})dtdx.$

2) $\Phi_2(\mathcal{A},\mathbf{u}(\mathcal{A})) = \int_0^T\int_\Omega\kappa(\mathcal{A},\mathbf{u}(\mathcal{A}))dtdx,$

$$\kappa(\mathcal{A},\mathbf{u}(\mathcal{A})) = \sum_{i\neq j}[a_{ij}(\sigma_{ii}-\sigma_{jj})^2 + b_{ij}\sigma_{ij}^2],\ a_{ij}>0,\ b_{ij}>0,$$
$$\sigma_{ij} \equiv \sigma_{ij}(\mathcal{A},\mathbf{u}(\mathcal{A}))(t) = A_{ijkl}(0)\epsilon_{kl}(\mathbf{u}(t)) + (A'_{ijkl} * \epsilon_{kl}(\mathbf{u}))(t).$$

The functional Φ_2 expresses the intensity of the shear stresses.

It can be verified using the standard methods that the Maximization problem $\mathcal{P}$ fulfils for all above mentioned choises of admissible sets and goal functions the conditions of the general theory and it has at least one solution $\mathcal{A}_* = \{A^*_{ijkl}\}$.

We continue with the finite element approximation of the Problem $\mathcal{P}$. We assume the polygonal region Ω divided regularly (see [4] for the details) by tetrahedrals $\{G_i\}$:

$$\bar{\Omega} = \bigcup_{i=1}^{I(h)} \bar{G}_i,\ G_i \cap G_j = \emptyset,\ i \neq j,\ h = \text{diam } G_i,\ i = 1, ..., I(h).$$

The division is consistent with the partition $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Let

$$V_h = \{\ \mathbf{v} \in V \cap C(\bar{\Omega})^3 :\ \mathbf{v}_{|G_i} \in \mathbf{P}_1\},$$

where $\mathbf{P}_1 \subset R^3$ is the space of vector polynomials of the first degree. Let us assume the admissible set $\mathcal{U}_{ad}$ defined in (62). In order to fulfil the regularity of coefficients $\{A_{ijkl}\}$ we shall consider the Hermitian interpolation with respect to the time variable. The method of Galerkin space-time discretization used in [10] can be used in final numerical algorithms.

For $\tau > 0$ we recall the division of the interval $[0, T]$ by

$$0 = t_0 < t_1 < ... < t_{N-1} < t_N = T,\ t_m = m\tau,\ m = 0, 1, ..., N(\tau)$$

and the approximation $u^{h\tau} \in C([0,T], V_h)$ of a weak solution $\mathbf{u}$ of (51)-(54) by

$$\mathbf{u}^{h\tau}(t) = \mathbf{u}^{h\tau}_{m-1} + \frac{t - t_{m-1}}{\tau}(\mathbf{u}^{h\tau}_m - \mathbf{u}^{h\tau}_{m-1}),\ t \in [t_{m-1}, t_m].$$

Approximate maximization problem $\mathcal{P}^{h\tau}$:

$$\mathcal{A}^{h\tau}_* = \arg \max_{\mathcal{A}\in\mathcal{U}^{h\tau}_{ad}} \Phi(\mathcal{A}, \mathbf{u}^{\mathbf{h}\tau}(\mathcal{A})),\ \mathcal{A} = \{A_{ijkl}\}$$

with the approximate admissible set of fourth-order tensor functions

$$\mathcal{U}_{ad}^{h\tau} = \{ \ \mathcal{A} \in \mathcal{U}_{ad} : \mathcal{A}(t) = \mathcal{A}_{m-1}^{\langle 0 \rangle} \phi_0 (\frac{t-t_{m-1}}{\tau}) + \\ \mathcal{A}_{m-1}^{\langle 1 \rangle} \phi_0 (\frac{t-t_{m-1}}{\tau}) + \mathcal{A}_{m}^{\langle 0 \rangle} \phi_1 (\frac{t-t_{m}}{\tau}) + \mathcal{A}_{m}^{\langle 1 \rangle} \phi_1 (\frac{t-t_{m}}{\tau}), \\ \mathcal{A}_{m}^{\langle 0 \rangle} = \mathcal{A}_{0}^{\langle 0 \rangle} + \tau \sum_{n=1}^{m} \mathcal{A}_{n}^{\langle 1 \rangle}, \ t_{m-1} \leq t \leq t_m, \ m = 1, ..., N(\tau) \},$$

$$\mathcal{A}_{m}^{\langle r \rangle} := \{ A_{ijkl,m}^{\langle r \rangle} \}, \ A_{ijkl,m}^{\langle r \rangle} \in U_h, \ r = 0, 1, \\ U_h = \{ \phi \in C(\bar{\Omega}) : \phi_{|G_n} \in P_1, \ n = 1, ... I(h) \}.$$

The Hermitian basic functions $\phi_0, \ \phi_1$ have the form

$$\phi_0(x) = \begin{cases} 1 - 3x^2 - 2x^3, & -1 \leq x \leq 0, \\ 1 - 3x^2 + 2x^3, & 0 \leq x \leq 1, \end{cases}$$

$$\phi_1(x) = \begin{cases} x + 2x^2 + x^3, & -1 \leq x \leq 0, \\ x - 2x^2 + x^3, & 0 \leq x \leq 1. \end{cases}$$

The discrete values of $\mathbf{u}^{h\tau}$ are determined by variational equations

$$\langle A_0^{\langle 0 \rangle} \mathbf{u}_0^{h\tau}, \mathbf{v} \rangle = \langle f_0^h, \mathbf{v} \rangle \ \forall \mathbf{v} \in V_h, \\ \langle A_0^{\langle 0 \rangle} \mathbf{u}_m^{h\tau} + \sum_{n=0}^{i-1} \tau A_{m-n}^{\langle 1 \rangle} \mathbf{u}_n^{h\tau}, \mathbf{v} \rangle = \langle f_m^h, \mathbf{v} \rangle \ \forall \mathbf{v} \in V_h, \\ m = 0, 1, ..., N(\tau) \tag{63}$$

with the operators $A_m^{\langle r \rangle} : V \to V^*$ defined by

$$\langle A_m^{\langle r \rangle} \mathbf{u}, \mathbf{v} \rangle = \int_\Omega A_{ijkl,m}^{\langle r \rangle}(\mathbf{x}) \epsilon_{ij}(\mathbf{u}) \epsilon_{kl}(\mathbf{v}) dx, \ \mathbf{u}, \ \mathbf{v} \in V,$$

and the finite element approximations f_m^h of the functionals $f(t_m) \in V^*$, $m = 1, ..., N(\tau)$ defined in (58).

Using the approach similar to the proof of the Theorem 2.2 the convergence of a subsequence of $\{\mathcal{A}^{h\tau}\}$ as $h \to 0+, \ \tau \to 0+$ can be verified.

Remark 3.2 The Maximization Problem (27) can be formulated also for the bending problem of a viscoelastic plate of variable thhickness and made of a long memory material. The deflections of the middle surface Ω are the elements of the Hilbert space

$$V = \{v \in H^2(\Omega) : v|_{\Gamma_0} = \frac{\partial v}{\partial \mathbf{n}}|_{\Gamma_0} = 0,\ v|_{\Gamma_1} = 0\},$$

if the part Γ_0 of the boundary $\partial\Omega$ is clamped and Γ_1 is simply supported.

The functionals $A(t) : V \to V^*$ are of the form

$$\langle A(t), v\rangle = \int_\Omega e^3(x) A_{ijkl}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_j \partial x_k} dx_1 dx_2$$

with the tensors $\{A_{ijkl}(x,t)\},\ i,j,k,l \in \{1,2\},\ x = (x_1, x_2)$ fulfilling the positive definiteness for $t = 0$. The variable thicknesses $e : \bar{\Omega} \to R$ can play the role of control parameters in a similar way as in [1], [2], where a minimization problem for a short memory material was investigated.

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Author's Addresses: Igor Bock, Department of Mathematics,
Faculty of Electrical Engineering and Information Technology,
Slovak University of Technology, 812 19 Bratislava, Slovakia.
email:bock@kmat.elf.stuba.sk.
Ján Lovíšek, Department of Mechanics, Faculty of Constructions,
Slovak University of Technology, 813 68 Bratislava, Slovakia.
email:lovisek@svf.stuba.sk