SUBDIRECT DECOMPOSITIONS OF LATTICE EFFECT ALGEBRAS

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Abstract. We prove a theorem about subdirect decompositions of lattice effect algebras. Further, we show how, under these decompositions, blocks, sets of sharp elements and centers of those effect algebras are decomposed. As an application we prove a statement about the existence of subadditive state on some block-finite effect algebras.
Key words: effect algebras, central elements, blocks, states, subdirect decompositions

1. Introduction and basic notions

In general, an effect algebra is a partial algebra with two constants $0, 1$ and a partial binary operation $\oplus$ (the orthogonal sum) satisfying very simple axioms introduced by Foulis and Bennett (1994). A model for an effect algebra is the standard effect algebra $\mathcal{E}(H)$ of positive self-adjoint operators dominated by the identity on the Hilbert space H.

Definition 1.1 Foulis and Bennett (1994)**.** A structure $(E; \oplus, 0, 1)$ is called an *effect-algebra* if 0, 1 are two distinguished elements and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,

(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,

(Eiii) for every $a \in E$ there exists a unique $b \in E$ such that $a \oplus b = 1$ (we put $a' = b$),

(Eiv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. In every effect algebra E we can define the partial operation $\ominus$ and the partial order $\leq$ by putting

$$a \leq b \text{ and } b \ominus a = c \text{ iff } a \oplus c \text{ is defined and } a \oplus c = b\,.$$

Since $a \oplus c = a \oplus d$ implies $c = d$, the $\ominus$ and the $\leq$ are well defined. If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete effect algebra*). For more details we refer the reader to [1] and the references given there.

Definition 1.2. Let $(E; \oplus, 0, 1)$ be an effect algebra. $Q \subseteq E$ is called a *sub-effect algebra of E* iff

(i) $1 \in Q$,

(ii) if $a, b, c \in E$ with $a \oplus b = c$ and out of a, b, c at least two elements are in Q then $a, b, c \in Q$.

Note that if Q is a sub-effect algebra of E then Q with inherited operation $\oplus$ is an effect algebra in its own right.

Definition 1.3. Let $(E; \oplus_E, 0_E, 1_E)$ and $(F; \oplus_F, 0_F, 1_F)$ be effect algebras. A bijective map $\varphi\colon E \to F$ is called an *isomorphism* if

(i) $\varphi(1_E) = 1_F$,

(ii) for all $a, b \in E$: $a \leq_E b'$ iff $\varphi(a) \leq_F \big(\varphi(b)\big)'$ in which case $\varphi(a \oplus_E b) = \varphi(a) \oplus_F \varphi(b)$.

We write $E \cong F$. Sometimes we identify E with $F = \varphi(E)$. If $\varphi\colon E \to F$ is an injection with properties (i) and (ii) then φ is called an *embedding*.

Note, that Kôpka (1992) introduced to effect algebras equivalent in some sense structures called D-posets, in which the operation of difference of fuzzy sets is the primary operation. For the connections we refer to [1] [10] and [12].

Finally, note that lattice effect algebras generalize orthomodular lattices and MV-algebras (including Boolean algebras), [1], [6], [7], [11].

For every central element z of a lattice effect algebra E, the interval $[0, z]$ with the $\oplus$ operation inherited from E and the new unity z is a lattice effect algebra in its own right. We are going to prove a statement about decompositions of E into subdirect products of such intervals $[0, z]$.

2. Compatibility, blocks and central elements

Recall that elements a, b of a lattice effect algebra $(E; \oplus, 0, 1)$ are called *compatible* (written $a \leftrightarrow b$) iff $a \vee b = a \oplus (b \ominus (a \wedge b))$ (see [11]). $P \subseteq E$ is a *set of pairwise compatible elements* if $a \leftrightarrow b$ for all $a, b \in P$. For $a \in E$ and $Q \subseteq E$ we will write $a \leftrightarrow Q$ iff $a \leftrightarrow q$ for all $q \in Q$. $M \subseteq E$ is called a *block* of E iff M is a maximal subset of pairwise compatible elements. Every block of a lattice effect algebra E is a sub-effect algebra and a sub-lattice of E and E is a union of its blocks (Riečanová, 2000). Lattice effect algebra with a unique block is called an *MV-effect algebra*. Every block of a lattice effect algebra is an MV-effect algebra in its own right. In [13] it was proved that *every block M of a lattice effect algebra E is closed with respect to all existing infima and suprema of subsets of M*. We say that M is a *full sub-lattice* of E.

An element z of an effect algebra E is called *central* if $x = (x \wedge z) \vee (x \wedge z')$ for all $x \in E$. The *center* $C(E)$ of E is the set of all central elements of E, (Greechie, Foulis and Pulmannová 1995). If E is a lattice effect algebra then $z \in E$ is central iff $z \wedge z' = 0$ and $z \leftrightarrow x$ for all $x \in E$, Riečanová (1999). Thus, in a lattice effect algebra E, $C(E) = B(E) \cap S(E)$, where $B(E) = \bigcap\{M \subseteq E \mid M \text{ is a block of } E\}$ is called a *compatibility center* of E and $S(E) = \{z \in E \mid z \wedge z' = 0\}$ is the set of all *sharp* elements of E. Evidently, $B(E) = \{x \in E \mid x \leftrightarrow y \text{ for all } y \in E\}$. In every lattice effect algebra E, $S(E)$ is an orthomodular lattice (Jenča, Riečanová, 1999), and $B(E)$ is an MV-effect algebra. Hence $C(E)$ is a Boolean algebra, [5]. Moreover, $S(E)$, $B(E)$, $C(E)$ are full sub-lattices of a lattice effect algebra E, which means that they are closed with respect to all infima and suprema existing in E, [15]. It follows that if E is a complete effect algebra then $S(E)$, $B(E)$, $C(E)$ and every block of E are also complete. Further, $B(E)$, $S(E)$ and $C(E)$ are sub-effect algebras of E, [8],[15].

In every lattice effect algebra E, if $z \in C(E)$ then the interval $[0,z]$ is a lattice effect algebra with the new unit z and the partial operation $\oplus$ inherited from E. It is because for $x, y \leq z$ with $x \oplus y$ defined in E we have $x \oplus y \leq z$ [5]. Further, $z = (x \oplus x') \wedge z = (x \wedge z) \oplus (x' \wedge z)$ for all $x \in E$. It follows that for $y \leq z$ we have $z = y \oplus (y' \wedge z)$ and hence $z \ominus y = y' \wedge d$, [17].

Our observations about subdirect products of lattice effect algebras are based on the following Theorem 2.1 from [8], reprinted also in the book [1], p. 107.

Lemma 2.1, (Jenča, Riečanová, 1999)**.** *Let E be a lattice effect algebra. Assume $b \in E$ and $A \subseteq E$ are such that $\bigvee A$ exists in E and $b \leftrightarrow a$ for all $a \in A$. Then*
(i) $b \leftrightarrow \bigvee A$
(ii) $\bigvee\{b \wedge a \mid a \in A\}$ *exists and equals* $b \wedge (\bigvee A)$.

Another useful statements will be about central elements.

Lemma 2.2, [17]**.** *Let u, v with $u \leq v'$ be elements of a lattice effect algebra E and $z \in C(E)$, then $(u \oplus v) \wedge z = (u \wedge z) \oplus (v \wedge z)$.*

Lemma 2.3. *Let E be a lattice effect algebra and $D \subseteq C(E)$ with (1) $\bigvee D = 1$ and (2) $d_1 \wedge d_2 = 0$ for all $d_1 \neq d_2,\ d_1, d_2 \in D$. Then the following conditions are equivalent for $u, v \in E$:*
(i) $u \leftrightarrow v$,
(ii) $\forall d \in D : u \wedge d \leftrightarrow v \wedge d$,
(iii) $\forall d_1, d_2 \in D : u \wedge d_1 \leftrightarrow v \wedge d_2$.

Proof. (i) $\Longrightarrow$ (ii) because $D \leftrightarrow E$. Further (ii) $\Longrightarrow$ (iii) as for every $d_1 \neq d_2,\ d_1, d_2 \in D$ we have $d_1 \leq d_2'$ which gives $u \wedge d_1 \leq d_1 \leq d_2' \leq d_2' \vee v' = (v \wedge d_2)'$ and hence $u \wedge d_1 \leftrightarrow v \wedge d_2$. By Lemma 2.1 we obtain (iii) $\Longrightarrow$ (i), as for each $d_1 \in D$ we have $v \wedge d_1 \leftrightarrow u = \bigvee\{u \wedge d \mid d \in D\}$ and hence $v = \{\bigvee\{v \wedge d \mid d \in D\} \leftrightarrow u$.

Lemma 2.4. *Let E be a lattice effect algebra and $d \in C(E)$. Then:*
(i) *If M is a block of E then $M \cap [0,d] = \{x \wedge d \mid x \in M\}$.*
(ii) *Elements $u, v \in [0,d]$ are compatible in $[0,d]$ iff $u \leftrightarrow v$ in E.*
(iii) *$M_d \subseteq [0,d]$ is a block od $[0,d]$ iff there exists a block M of E with $M \cap [0,d] = M_d$.*
(iv) $S([0,d]) = \{x \wedge d \mid x \in S(E)\} = S(E) \cap [0,d]$.
(v) $B([0,d]) = \{x \wedge d \mid x \in B(E)\} = B(E) \cap [0,d]$.
(vi) $C([0,d]) = \{x \wedge d \mid x \in C(E)\} = C(E) \cap [0,d]$.

Proof.
(i) Let $x \in M$. Since $d \leftrightarrow E$ and $x \leftrightarrow M$ we obtain $x \wedge d \leftrightarrow M$, which, by the maximality of blocks, gives $x \wedge d \in M$. Conversely, $y \in M \cap [0,d]$ implies $y = y \wedge d \in M$.
(ii) This follows from the fact that $[0,d]$ is a sublattice of E which is closed with respect to $\oplus$.
(iii) If M_d is a block of $[0,d]$ then, by (ii), M_d is a pairwise compatible set in E and hence by [13] there exists a block M of E such that $M_d \subseteq M$. Hence $M_d \subseteq M \cap [0,d]$, which, by (i) and (ii) and the maximality of blocks, gives $M_d = M \cap [0,d]$. Conversely, if M is a block of E then $M \cap [0,d]$ is a pairwise compatible set in $[0,d]$, by (i) and Lemma 2.3. Further, if $y \in [0,d]$ such that

$y \leftrightarrow M \cap [0, d]$ then $y = y \wedge d \leftrightarrow x \wedge d$ for all $x \in M$ by (i). Further, for all $x \in M$ and $d_1 \neq d$, $d_1 \in D$ we have $y \leq d \leq d_1' \leq d_1' \vee x' = (d_1 \wedge x)'$ and hence $y \leftrightarrow d_1 \wedge x$. By Lemma 2.1 we obtain $y \leftrightarrow x$ for all $x \in M$. By the maximality of blocks we obtain $y \in M$, hence $y \in M \cap [0, d]$. We conclude that $M \cap [0, d]$ is a block od $[0, d]$.

(iv) If $x \in S(E) \cap [0, d]$ then $x \wedge x' = 0$ and $x \leq d$ which gives $(x \wedge d) \wedge (x' \wedge d) \leq x \wedge x' = 0$ and hence $x = x \wedge d \in S[0, d]$. If $x \in S([0, d])$ then $x = x \wedge d$ and $x \wedge x' = (x \wedge d) \wedge x' = x \wedge (x' \wedge d) = 0$ which gives $x \in S(E) \cap [0, d]$.

(v) $B(E) \cap [0, d] = \bigcap\{M \cap [0, d] \mid M$ is a block of $E\} = \bigcap\{M_d \mid M_d$ is a block of $[0, d]\} = B([0, d])$. Moreover, $x \in B(E) \cap [0, d]$ implies $x = x \wedge d$. Conversely, for $x \in B(E)$ we have $x \wedge d \leftrightarrow E$ and hence $x \wedge d \in B([0, d])$.

(vi) $C(E) \cap [0, d] = B(E) \cap S(E) \cap [0, d] = B([0, d]) \cap S([0, d])$, by (iv) and (v). Hence $C([0, d]) = C(E) \cap [0, d]$. Moreover, $x \in C(E) \cap [0, d]$ implies $x = x \wedge d$. Conversely, $x \in C(E)$ implies $x \wedge d \leftrightarrow E$ and $(x \wedge d) \wedge (x' \wedge d) = 0$ which gives $x \wedge d \in C([0, d])$.

3. Subdirect decompositions of lattice effect algebras

A *direct product* of a family of effect algebras $\{E_\varkappa \mid \varkappa \in H\}$, $H \neq \emptyset$ is the Cartesian product $\prod\{E_\varkappa \mid \varkappa \in H\}$ with componentwise defined operations $\hat{\oplus}, \hat{0}, \hat{1}$, which means that $(a_\varkappa)_{\varkappa \in H} \hat{\oplus} (b_\varkappa)_{\varkappa \in H} = (a_\varkappa \oplus_\varkappa b_\varkappa)_{\varkappa \in H}$ iff $a_\varkappa \oplus_\varkappa b_\varkappa$ is defined in $E_\varkappa$, for all $\varkappa \in H$. Further, $\hat{0} = (0_\varkappa)_{\varkappa \in H}$ and $\hat{1} = (1_\varkappa)_{\varkappa \in H}$. Evidently, for each $\varkappa_i \in H$, the natural projection $\mathrm{pr}_{\varkappa_i}$ of $\prod\{E_\varkappa \mid \varkappa \in H\}$ onto $E_{\varkappa_i}$ is a homomorphism. $\prod\{E_\varkappa \mid \varkappa \in H\}$ is a *direct product decomposition* of an effect algebra E if there is an isomorphism $\varphi \colon E \to \prod\{E_\varkappa \mid \varkappa \in H\}$.

A *subdirect product* of a family $\{E_\varkappa \mid \varkappa \in H\}$ of effect algebras, $H \neq \emptyset$ is a sub-effect algebra Q of the direct product $\prod\{E_\varkappa \mid \varkappa \in H\}$ such that each restriction $pr_{\varkappa_i}|_Q$, $\varkappa_i \in H$, of the natural projection $\mathrm{pr}_{\varkappa_i}$ to Q, is onto $E_{\varkappa_i}$. Q is a *subdirect product decomposition* of an effect algebra E if there exists an isomorphism $\varphi \colon E \to Q$.

Theorem 3.1. *Let $(E; \oplus, 0, 1)$ be a lattice effect algebra and $D \subseteq C(E)$ with (1) $\bigvee D = 1$ and (2) $d_1 \wedge d_2 = 0$ for all $d_1 \neq d_2$; $d_1, d_2 \in D$. Then:*

(i) *E is isomorphic to a subdirect product of the family of effect algebras $\{[0, d] \mid d \in D\}$.*

(ii) *Up to isomorphism, E is a sub-lattice of $\hat{E} = \prod\{[0, d] \mid d \in D\}$.*

(iii) *If, moreover, E is complete or D is finite then $E \cong \hat{E}$.*

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(i) By Lemma 2.1, for every $u \in E$ we have $u = u \wedge (\bigvee D) = \bigvee\{u \wedge d \mid d \in D\}$. Let $Q = \{(u \wedge d)_{d \in D} \mid u \in E\}$ and the map $\varphi : E \to Q$ be defined by the formula

$$\varphi(u) = (u \wedge d)_{d \in D}, \text{ for all } u \in E.$$

Let us show that Q is a sub-effect algebra of $\hat{E} = \prod\{[0, d] \mid d \in D\}$ and φ is an isomorphism. Because operations $\hat{\oplus}$, $\hat{0}$ and $\hat{1}$ in $\hat{E}$ are defined componentwise, we obtain that for every $x = (u \wedge d)_{d \in D} \in Q$ we have $x' = (u' \wedge d)_{d \in D} \in Q$ and $x \hat{\oplus} x' = ((u \wedge d) \oplus (u' \wedge d))_{d \in D} = ((u \oplus u') \wedge d)_{d \in D} = (1 \wedge d)_{d \in D} = \hat{1} \in Q$. Thus

$\varphi(u') = (\varphi(u))'$. Because the partial order in $\hat{E}$ is derived from $\hat{\oplus}$, hence it is also defined componentwise, we have for $y = (v \wedge d)_{d\in D} \in Q$ that $y \leq x'$ iff $v \leq u'$ iff $\varphi(v) \leq (\varphi(u))'$ and $\varphi(u)\hat{\oplus}\varphi(v) = x\hat{\oplus}y = (u \wedge d)_{d\in D}\hat{\oplus}(v \wedge d)_{d\in D} = ((u \wedge d) \oplus (v \wedge d))_{d\in D} = ((u \oplus v) \wedge d)_{d\in D} = \varphi(u \oplus v) \in Q$. Further, we can easily see that for $u, v \in E$ we have $\varphi(u) = \varphi(v)$ iff $u \wedge d = v \wedge d$ for all $d \in D$ iff $u = v$ by Lemma 2.1. Moreover, the restriction $pr_d|_Q$ of the natural projection pr_d to Q is onto $[0, d]$, hence Q is a subdirect product of $\hat{E}$

(ii) Because, up to isomorphism $[0, d] \subseteq E \subseteq \hat{E}$, for every $x = (u \wedge d)_{d\in D} \in Q$ we have $\bigvee\{u \wedge d \mid d \in D\} = u \in E$, we can conclude that, up to isomorphism E is a sublattice of $\hat{E}$, by Lemma 2.1.

(iii) If D is a finite set or $\hat{E}$ is a complete lattice then for every $(u_d)_{d\in D} \in \hat{E}$ we have $\bigvee\{u_d \mid d \in D\} = u \in E$.

Corollary 3.2. *Under the assumptions of Theorem 1, if, moreover E is complete or D is finite then:*

(i) *M is a block of E iff $M \cong \prod\{M_d \mid d \in D,\ M_d$ is a block of $E\} = \prod\{M \cap [0, d] \mid d \in D\}$.*

(ii) $S(E) \cong \prod\{S([0, d]) \mid d \in D\}$.

(iii) $B(E) \cong \prod\{B([0, d]) \mid d \in D\}$.

(iv) $C(E) \cong \prod\{C([0, d]) \mid d \in D\}$.

Proof. We have shown in the proof of Theorem 3.1 that the map $\varphi : E \to \prod\{[0, d] \mid d \in D\}$ defined by $\varphi(u) = (u \wedge d)_{d\in D}$, for all $u \in E$, is an isomorphism.

(i) The statement follows by Lemmas 1 and 2 and the maximality of blocks, under which, the restriction $\varphi|_M$ is the wanted isomorphism.

(ii) By Lemma 2.4, (iv), the restriction $\varphi|_{S(E)}$ is the wanted isomorphism.

(iii) By Lemma 2.4, (v), the restriction $\varphi|_{B(E)}$ is the wanted isomorphism.

(iv) By Lemma 2.4, (vi), the restriction $\varphi|_{C(E)}$ is the wanted isomorphism.

In the next theorem and subsequently we use the following notation: For an arbitrary mapping $\varphi\colon X \to Y$ and $A \subseteq Y$, the *inverse image* of A denoted by $\varphi^{-1}(A)$ is the set

$$\varphi^{-1}(A) = \{x \in X \mid f(x) \in A\} \subseteq X\,.$$

Clearly, if φ is a bijection this is identical with the image of A by the inverse mapping φ^{-1}.

Theorem 3.3. *Under the assumptions of Theorem 3.1, if φ_D is the embedding of E into $\prod\{[0, d] \mid d \in D\}$ defined for all $x \in E$ by $\varphi_D(x) = (x \wedge d)_{d\in D}$, then*

(i) *M is a block of E iff there are blocks M_d of $[0, d]$, $d \in D$ such that $M = \varphi_D^{-1}\big(\prod\{M_d \mid d \in D\}\big)$.*

(ii) $S(E) = \varphi_D^{-1}\big(\prod\{S([0, d]) \mid d \in D\}\big)$,

(iii) $B(E) = \varphi_D^{-1}\big(\prod\{B([0, d]) \mid d \in D\}\big)$,

(iv) $C(E) = \varphi_D^{-1}\big(\prod\{C([0, d]) \mid d \in D\}\big)$.

Proof. (i) By Lemmas 2.3 and 2.4, M is a maximal pairwise compatible set of elements of E iff for each $d \in D$, $\{u \wedge d \mid u \in M\}$ is a maximal pairwise compatible set in $[0, d]$. Thus the statement follows by the definition of φ_D and blocks.

(ii)–(iv) These follow by Lemma 2.4 and the definition of φ_D.

Note that Theorem 3.3 and Lemma 2.4 actually show that every block M of E can be subdirectly decomposed by family $\{M_d \mid d \in D\}$ of blocks of $[0, d]$. Similar statements are true for the set of all sharp elements of E, compatibility center of E and center of E.

It is known that on every MV-algebra there exists a state. Moreover, every state on an MV-algebra is subadditive [7]. Unfortunately, there are even finite (lattice) effect algebras and orthomodular lattices admitting no states [18]. Some positive results were given e.g. in [19]. Further positive results we can obtain as applications of decompositions of effect algebras.

Definition 3.4. A map $\omega : E \to [0, 1] \subseteq R$ is called a *state on the effect algebra* E if (i) $\omega(1) = 1$ and (ii) if $a, b \in E$ with $a \leq b'$ then $\omega(a \oplus b) = \omega(a) + \omega(b)$. If, moreover, E is lattice ordered then ω is called *subadditive* if $\omega(a \vee b) \leq \omega(a) + \omega(b)$, for all $a, b \in E$.

Note that a state on a lattice effect algebra need not to be subadditive, [19].

Lemma 3.5. *Let E be a lattice effect algebra. A (subadditive) state on E exists iff there is a nonzero $d \in C(E)$ such that there exists a (subadditive) state on $[0, d]$.*

Proof. If $d \in C(E)$, $d \neq 0$ then for $x, y \in E$ with $x \leq y'$ we have $(x \oplus y) \wedge d = (x \wedge d) \oplus (y \wedge d)$ by Lemma 2.4. Moreover, for every $u, v \in E$, by Lemma 2.3 we have $(u \vee v) \wedge d = (u \wedge d) \vee (v \wedge d)$. It follows that if $m : [0, d] \to [0, 1] \subseteq R$ is a (subadditive) state on $[0, d]$ then $\omega : E \to [0, 1] \subseteq R$ defined for all $x \in E$ by $\omega(x) = m(x \wedge d)$ is a (subadditive) state on E. The converse is clear, as $1 \in C(E)$.

Theorem 3.6. *Let E be a lattice effect algebra with exactly n blocks and the center $C(E) \neq \{0, 1\}$. If n is a prime number then:*

(i) *For every $d \in C(E) \setminus \{0, 1\}$, at least one of the effect algebras $[0, d]$ and $[0, d']$ is an MV-effect algebra.*

(ii) *There exists a subadditive state on E.*

Proof.

(i) Let $d \in C(E) \setminus \{0, 1\}$. By Theorem 3.1, $E \cong [0, d] \times [0, d']$. Moreover, by Corollary 3.2, (i) we have $n = k_1 \cdot k_2$ under which k_1 and k_2 are numbers of blocks of $[0, d]$ and $[0, d']$ respectively. As n is a prime number, we conclude that at least one of k_1 and k_2 equals to 1.

(ii) Assume that $[0, d]$ is an MV-effect algebra. Then there exists a subadditive state m on $[0, d]$, as $[0, d]$ can be organized into an MV-algebra. By Lemma 3.5, there exists a subadditive state on E.

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