CENTRAL ELEMENTS, BLOCKS AND SHARP ELEMENTS OF LATTICE EFFECT ALGEBRAS

Gejza Jenča, Ivica Marinová, Zdenka Riečanová

Abstract. For every central element z of a lattice effect algebra E, the interval $[0, z]$ with the $\oplus$ operation inherited from E and the new unity z is a lattice effect algebra in its own right. We show connections between blocks, sharp elements and central elements of $[0, z]$ and those of E. We prove that except for central elements, the intervals $[0, z]$ are closed with respect to the $\oplus$-operation also for all sharp elements

1. Introduction and basic definitions

Effect algebras have been introduced by Foulis and Bennet [2] as an algebraic structure providing an instrument for studying quantum effects that may be unsharp. Kôpka [10] introduced a D-poset of fuzzy sets in which the operation of difference of fuzzy sets is the primary operation. For the connection between effect algebras and D-posets we refer to [1] and [12].

In recent years effect algebras [2] or equivalent in some sense D-posets [10], [11] have been studied as carriers of states or probability measures in the quantum or fuzzy probability theory.

Lattice effect algebras generalize orthomodular lattices and MV-algebras (including Boolean algebras), [1], [11]. Moreover, every lattice effect algebra E is a union of MV-effect algebras (effect algebras which can be organized into MV-algebras), the set of all sharp elements of E is an orthomodular lattice and the center of E is a Boolean algebra.

Definition 1.1. A structure $(E; \oplus, 0, 1)$ is called an *effect-algebra* if 0, 1 are two distinguished elements and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,
(Eiii) for every $a \in E$ there exists a unique $b \in E$ such that $a \oplus b = 1$ (we put $a' = b$),
(Eiv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. In every effect algebra E we can define the partial operation $\ominus$ and the partial order $\leq$ by putting

$$a \leq b \text{ and } b \ominus a = c \text{ iff } a \oplus c \text{ is defined and } a \oplus c = b\,.$$

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Since $a \oplus c = a \oplus d$ implies $c = d$, the $\ominus$ and the $\leq$ are well defined. If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete effect algebra*). For more details we refer the reader to [1] and the references given there.

Definition 1.2. Let $(E; \oplus, 0, 1)$ be an effect algebra. $Q \subseteq E$ is called a *sub-effect algebra of E* iff

(i) $1 \in Q$,
(ii) if $a, b, c \in E$ with $a \oplus b = c$ and out of a, b, c at least two elements are in Q then $a, b, c \in Q$.

Note that if Q is a sub-effect algebra of E then Q with inherited operation $\oplus$ is an effect algebra in its own right.

Recall that elements a, b of a lattice effect algebra $(E; \oplus, 0, 1)$ are called *compatible* (written $a \leftrightarrow b$) iff $a \vee b = a \oplus (b \ominus (a \wedge b))$ (see [11]). $P \subseteq E$ is a *set of pairwise compatible elements* if $a \leftrightarrow b$ for all $a, b \in P$. For $a \in E$ and $Q \subseteq E$ we will write $a \leftrightarrow Q$ iff $a \leftrightarrow q$ for all $q \in Q$. $M \subseteq E$ is called a *block* of E iff M is a maximal subset of pairwise compatible elements. Every block of a lattice effect algebra E is a sub-effect algebra and a sub-lattice of E and E is a union of its blocks (see [13]). Lattice effect algebra with a unique block is called an *MV-effect algebra*. Every block of a lattice effect algebra is an MV-effect algebra in its own right. In [13] it was proved that *every block M of a lattice effect algebra E is closed with respect to all existing infima and suprema of subsets of M*. We say that M is a *full sub-lattice* of E.

A lattice effect algebra E is a *horizontal sum* of blocks if $A \cap B = \{0, 1\}$ holds for every pair of its blocks A and B.

A nonzero element a of an effect algebra E is called an *atom* if $0 \leq b < a$ implies $b = 0$. E is called atomic if for every nonzero element $x \in E$ there is an atom $a \in E$ with $a \leq x$.

An effect algebra E is called *Archimedean* if for no nonzero element $e \in E$, $ne = e \oplus e \oplus \cdots \oplus e$ (n times) exists for all positive integer n. We write $\mathrm{ord}(e) = n_e \in N$ if n_e is the greatest integer such that $n_e e$ exists in E. Every complete effect algebra is Archimedean, [20].

Definition 1.3. Let $(E; \oplus_E, 0_E, 1_E)$ and $(F; \oplus_F, 0_F, 1_F)$ be effect algebras. A bijective map $\varphi\colon E \to F$ is called an *isomorphism* if

(i) $\varphi(1_E) = 1_F$,
(ii) for all $a, b \in E$: $a \leq b'$ iff $\varphi(a) \leq \big(\varphi(b)\big)'$ in which case $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$.

We write $E \cong F$. Sometimes we identify E with $F = \varphi(E)$.

2. Blocks of lattice effect algebras and of sets of sharp elements

Assume that $(L; \vee, \wedge, ^{\perp}, 0, 1)$ is an orthomodular lattice [9]. Then L becomes a lattice effect algebra if we define that $a \oplus b$ exists iff $a \leq b^{\perp}$ and then $a \oplus b = a \vee b$, [1], [10]. In an orthomodular lattice L, two elements $a, b \in L$ are compatible iff $a = (a \wedge b) \vee (a \wedge b^{\perp})$ iff $b = (a \vee b) \vee (a' \wedge b)$, [9]. If L is the effect algebra derived from the orthomodular lattice L then for $b \in L$ we have $1 = b \vee b^{\perp} = b \oplus b^{\perp}$, since $b \leq (b^{\perp})^{\perp}$ and $b \wedge b^{\perp} = 0$. Thus $b^{\perp} = 1 \ominus b = b'$ for all $b \in L$. Further, elements a, b of the orthomodular lattice L are compatible iff $a = (a \wedge b) \vee (a \wedge b^{\perp})$ iff $b = (a \wedge b) \vee (a^{\perp} \wedge b)$ which gives $a = (a \wedge b) \oplus (a \wedge b')$

and $b = (b \wedge a) \oplus (a' \wedge b)$. It follows that $a \wedge b' = a \ominus (a \wedge b)$ and $(a \wedge b') = b \ominus (a \wedge b)$ which gives $(a \wedge b') \vee (a' \wedge b) = (a \wedge b') \oplus (a' \wedge b) = (a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b))$ and hence $a \vee b = (a \wedge b) \oplus (a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b)) = a \oplus (b \ominus (a \wedge b))$ which gives $a \leftrightarrow b$. Thus *$a, b \in L$ are compatible in an orthomodular lattice L iff a, b are compatible in the derived effect algebra L.*

In [8] has ben shown that in every lattice effect algebra E the subset $S(E) = \{x \in E \mid x \wedge x' = 0\}$ of all sharp elements of E is an orthomodular lattice being a sub-lattice and a sub-effect algebra of E.

According to the above proved statement on the compatible pairs in an orthomodular lattice and the derived effect algebra, it is clear that *for every block M of a lattice effect algebra E the intersection $M \cap S(E)$ is a set of pairwise compatible elements* in the orthomodular lattice $S(E)$ and the derived lattice effect algebra $S(E)$ as well. In spite of this fact *$M \cap S(E)$ in general is not a block of $S(E)$*, as the following example shows.

Example 2.1. Let $E = \{0, a, a', b, 1\}$ where $a \oplus a' = 1 = b \oplus b$. Evidently $S(E) = \{0, a, a', 1\}$. Further $\{0, b, 1\}$ is a block of E, while $M \cap S(E) = \{0, 1\}$ is not a block of $S(E)$, because $S(E)$ is a Boolean algebra.

3. Central elements in a lattice effect algebras

An element z of an effect algebra E is called *central* if $x = (x \wedge z) \vee (x \wedge z')$ for all $x \in E$. The *center* $C(E)$ of E is the set of all central elements of E, [5]. If E is a lattice effect algebra then $z \in E$ is central iff $z \wedge z' = 0$ and $z \leftrightarrow x$ for all $x \in E$, [12]. Thus in a lattice effect algebra E, $C(E) = B(E) \cap S(E)$, where $B(E) = \bigcap\{M \subseteq E \mid M \text{ is a block of } E\}$ is called a *compatibility center* of E and $S(E) = \{z \in E \in z \wedge z' = 0\}$ is the set of all *sharp* elements of E. Evidently, $B(E) = \{x \in E \mid x \leftrightarrow y \text{ for all } y \in E\}$. In every lattice effect algebra E, $S(E)$ is an orthomodular lattice, [8] and $B(E)$ is an MV-effect algebra. Hence $C(E)$ is a Boolean algebra, [5]. Moreover, $S(E)$, $B(E)$, $C(E)$ and all blocks of E are full sub-lattices of a lattice effect algebra E, which means that they are closed with respect to all infima and suprema existing in E, [15]. It follows that if E is a complete effect algebra then $S(E)$, $B(E)$, $C(E)$ and every block of E are also complete. Further, $B(E)$, $S(E)$ and $C(E)$ are sub-effect algebras of E, [8],[15].

Assume that $z \in E$ is a central element of the lattice effect algebra E. Then the interval $[0, z]$ is a lattice effect algebra with the unit z and the partial operation $\oplus$ inherited from E. It is because for $x, y \leq z$ with $x \oplus y$ defined in E we have $x \oplus y \leq z$, [5]. Further, $z = (x \oplus x') \wedge z = (x \wedge z) \oplus (x' \wedge z)$ for all $x \in E$. It follows that for $y \leq z$ we have $z = y \oplus (y' \wedge z)$ and hence $z \ominus y = y' \wedge z$, [17].

By the definition of central element and the properties mentioned above it is clear that $z \in E$ is central iff $E \cong [0, z] \times [0, z']$ where $[0, z] \times [0, z']$ is a cartesian product with "componentwise" defined operations, which means that for $(a, b), (c, d) \in [0, z] \times [0, z']$ we have $(a, b) \oplus (c, d) = (a \oplus c, b \oplus d)$ iff $a \leq c'$ and $b \leq d'$ and $(0, 0)$ is the zero and $(1, 1)$ is the unit in the product, [5].

Theorem 3.1. *If z is a central element of a lattice effect algebra E then $S([0, z]) = S(E) \cap [0, z]$.*

Proof. For every $y \leq z$ we have $z = (y \oplus y') \wedge z = (y \wedge z) \oplus (y' \wedge z) = y \oplus (y' \wedge z)$, [15].

It follows that $y \in S[0,z]$ iff $y \wedge (y' \wedge z) = 0$ iff $y \wedge y' = 0$, as $y \wedge z = y$. This proves the theorem.

Theorem 3.2. *Let E be a lattice effect algebra, M be a block of E and $a \in E$.*
- (i) *If a is an atom of M then a is an atom of E.*
- (ii) *If a is an atom of E and $a \in C(E)$ then $[a,1] \cap [0,a'] = \emptyset$ and $E = [a,1] \cup [0,a']$.*
- (iii) *$C(M) = S(M)$, $B(M) = M$ and $S(M) = S(E) \cap M$.*

Proof. (i) Let $0 < b < a$. Then for every $x \in M$ we have $a \leftrightarrow x$ and hence $x \vee a = a \oplus (x \ominus (a \wedge x)$. It follows that either $a \leq x$ or $x \vee a = a \oplus x$ which gives $a \leq x'$. Thus either $b \leq x$ or $b \leq x'$, which implies that $b \leftrightarrow x$, and by maximality of blocks we obtain that $b \in M$, a contradiction.

(ii) Since $C(E) = B(E) \cap S(E)$ we have $a \wedge a' = 0$ and hence $[a,1] \cap [0,a'] = \emptyset$. Moreover, $x \leftrightarrow a$ for all $x \in E$, which implies that $a \leq x$ or $a \leq x'$. Thus $E = [a,1] \cup [0,a']$.

(iii) $B(M) = M$ by the definition of blocks. Further, if $x \in M$ then $x' \in M$ and $x \wedge x' \in M$. Hence $S(M) = S(E) \cap M$ and $C(M) = B(M) \cap S(M) = M \cap S(M) = S(M)$.

Theorem 3.3. *For every block M and every central element z of a lattice effect algebra E the intersection $M \cap [0,z]$ is a block of $[0,z]$.*

Proof. For every $x, y \in M$ we have $x \leftrightarrow y \wedge z$ since $z \leftrightarrow E$ and $x \leftrightarrow y$. By maximality of blocks we obtain that $M_z = \{x \wedge z \mid y \in M\} \subseteq M$, hence $M_z = M \cap [0,z]$. Assume that $u \in E$ with $u \leq z$ and $u \leftrightarrow M_z$. Then $u \leftrightarrow x$ for all $x \in M$ because $u \leq z \leq v'$ for every $v \leq z'$ and $u \leftrightarrow x \wedge z$ which gives $u \leftrightarrow x = (x \wedge z) \oplus (x \wedge z')$. It follows that $u \in M \cap [0,z]$. By maximality of blocks, we conclude that $M \cap [0,z]$ is a block of $[0,z]$.

Corollary 3.4. *For every central element z of a lattice effect algebra E*
$$B([0,z]) = B(E) \cap [0,z] \text{ and } C([0,z]) = C(E) \cap [0,z].$$

Proof. By Theorem 3.3 $B(E) \cap [0,z] = \bigcap\{M \cap [0,z] \mid M \text{ is a block of } E\} \supseteq B([0,z])$. Conversely, for $y \in B(E) \cap [0,z]$ we have $y \in [0,z]$ and $y \leftrightarrow E$ which gives $y \leftrightarrow [0,z]$ and hence $y \in B([0,z])$. We conclude that $B(E) \cap [0,z] = B([0,z])$. It follows that $C([0,z]) = B([0,z]) \cap S([0,z]) = B(E) \cap [0,z] \cap S(E) \cap [0,z] = C(E) \cap [0,z]$.

Theorem 3.5. *Let z be a central element of a lattice effect algebra E and M be a block of E. Then*
- (i) $M \cong (M \cap [0,z]) \times (M \cap [0,z'])$,
- (ii) $S(E) \cong S([0,z]) \times S([0,z'])$,
- (iii) $B(E) \cong B([0,z]) \times B([0,z'])$,
- (iv) $C(E) \cong C([0,z]) \times C([0,z'])$,

Proof. (i) $z \in C(E)$, therefore $E \cong [0,z] \cap [0,z']$, [5]. Since $\oplus$, zero and unit in $[0,z] \times [0,z']$ are defined componentwise, $\leq$, $\vee$ and $\wedge$ are defined componentwise as well. It follows that $(a,b),(c,d) \in [0,z] \times [0,z']$ are compatible iff $a \leftrightarrow c$ and $b \leftrightarrow d$. Thus every maximal subset M of pairwise compatible elements of the product is a product of maximal subsets M_1 and M_2 of pairwise compatible elements in $[0,z]$ and $[0,z']$, respectively. Combining this with the fact that $M \cap [0,z]$ is a block of $[0,z]$ and $M \cap [0,z']$ is a block of $[0,z']$ we obtain the proof of (i).

(ii) Since $(a,b)\wedge(a',b')=0$ iff $a\wedge a'=0$ and $b\wedge b'=0$ iff $a\in S([0,z])$ and $b\in S([0,z'])$, we obtain that $S(E)\cong S([0,z])\times S([0,z'])$.

(iii) is a consequence of (i).

(iv) Since $C(E)=B(E)\cap S(E)$ for every lattice effect algebra E, we conclude that (iv) is a consequence of (ii) and (iii).

4 Sharp elements of a lattice effect algebras

Assume that E is a lattice effect algebra. In general, if for elements $a,b\leq c$ the sum $a\oplus b$ exists then the inequality $a\oplus b\leq c$ need not hold.

Example 4.1. Let $E=\{0,a,2a,3a,1=4a\}$ be a chain. Then $a,2a\leq 2a$ but $a\oplus 2a\not\leq 2a$.

Theorem 4.2. *For every element c of a lattice effect algebra E the following conditions are equivalent:*

(i) $c\in S(E)$.

(ii) *For all $a,b\in[0,c]$ with $a\leq b'$ the $a\oplus b\leq c$.*

Proof. (i) $\Longrightarrow$ (ii): since $a\wedge b\leq a\vee b\leq c$ and every chain in E is in a block of E, there is a block M of E such that $\{a\wedge b, a\vee b, c\}\subseteq M$. By Theorem 3.2, $S(E)\cap M=C(M)$, hence c is central in M. Further, $a\oplus b=(a\wedge b)\oplus(a\vee b)$. Because $a\wedge b, a\vee b\leq c$ we conclude that $(a\wedge b)\oplus(a\vee b)=a\oplus b\leq c$.

(ii) $\Longrightarrow$ (i): see [5].

Theorem 4.3. *For every element z of an Archimedean lattice effect algebra E the following conditions are equivalent:*

(i) $z\in S(E)$.

(ii) *If $x\leq z$ and $nx=x\oplus\cdots\oplus x$ (n times) exists then $nx\leq z$.*

Proof. (i) $\Longrightarrow$ (ii): This follows by induction using Theorem 4.2.

(ii) $\Longrightarrow$ (i): Put $x=z\wedge z'$. By (ii) we have $nx\leq z\leq z\vee z'=(z\wedge z')'=x'$ for every defined nx. It follows that $x=0$, because otherwise $(n+1)x$ is defined for every n with defined nx. This contradicts to the Archimedean property of E.

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Department of Mathematics,
Faculty of Electrical Engineering and Information Technology,
Slovak Technical University,
Ilkovičova 3,
812 19 Bratislava, Slovak Republic

E-mail address: zriecan@elf.stuba.sk