BOOLEAN ALGEBRAS R-GENERATED BY MV-EFFECT ALGEBRAS

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Abstract. We prove that every MV-effect algebra M is, as an effect algebra, a homomorphic image of its R-generated Boolean algebra. We characterize central elements of M in terms of the constructed homomorphism.

1. Definitions and basic relationships

An *effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation $\oplus$ and two nullary operations $0, 1$ satisfying the following conditions.

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$.
(E4) If $a \oplus 1$ exists, then $a = 0$

Effect algebras were introduced by Foulis and Bennett in their paper [5]. Independently, Kôpka and Chovanec introduced an essentially equivalent structure called *D-poset* (see [10]). Another equivalent structure, called *weak orthoalgebras* was introduced by Giuntini and Greuling in [6]. We refer to [4] for more information on effect algebras and similar algebraic structures.

For brevity, we denote an effect algebra $(E; \oplus, 0, 1)$ by E. In an effect algebra E, we write $a \leq b$ iff there is $c \in E$ such that $a \oplus c = b$. It is easy to check that every effect algebra is cancellative, thus $\leq$ is a partial order on E. In this partial order, 0 is the least and 1 is the greatest element of E. Moreover, it is possible to introduce a new partial operation $\ominus$; $b \ominus a$ is defined iff $a \leq b$ and then $a \oplus (b \ominus a) = b$. It can be proved that $a \oplus b$ is defined iff $a \leq b'$ iff $b \leq a'$. Therefore, it is usual to denote the domain of $\oplus$ by $\perp$. If $a \perp b$, we say that a and b are *orthogonal*.

Let E_1, E_2 be effect algebras. A mapping $\phi : E_1 \mapsto E_2$ is called a *homomorphism* iff $\phi(1) = 1$ and $a \perp b$ implies that $\phi(a) \perp \phi(b)$ and then $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. A homomorphism ϕ is an *isomorphism* iff ϕ is bijective and ϕ^{-1} is a homomorphism. Note that even if both E_1 and E_2 are lattice ordered, a homomorphism need not to preserve joins and meets.

An *MV-algebra* (c.f. [2], [12]) is a $(2, 1, 0)$-type algebra $(M; \boxplus, \neg, 0)$, such that $\boxplus$ satisfying the identities $(x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$, $x \boxplus z = y \boxplus x$, $x \boxplus 0 = x$, $\neg\neg x = x$, $x \boxplus \neg 0 = \neg 0$ and

$$x \boxplus \neg(x \boxplus \neg y) = y \boxplus \neg(y \boxplus \neg x).$$

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An *MV-effect algebra* is a lattice ordered effect algebra M in which, for all $a, b \in M$, $(a \vee b) \ominus a = b \ominus (a \wedge b)$. It is proved in [3] that there is a natural, one-to one correspondence between MV-effect algebras and MV-algebras given by the following rules. Let $(M, \oplus, 0, 1)$ be an MV-effect algebra. Let $\boxplus$ be a total operation given by $x \boxplus y = x \oplus (x' \wedge y)$. Then $(M, \boxplus, 0, 1)$ is an MV-algebra. Similarly, let $(M, \boxplus, 0, 1)$ be an MV-algebra. Restrict the operation $\boxplus$ to the pairs (x, y) satisfying $x \leq y'$ and call the new partial operation $\oplus$. Then $(M, \oplus, 0, 1)$ is an MV-effect algebra.

Among lattice ordered effect algebras, MV-effect algebras can be characterized in a variety of ways. Three of them are given in the following proposition.

Proposition 1. [1], [3] *Let E be a lattice ordered effect algebra. The following are equivalent*

(a) *E is an MV-effect algebra.*
(b) *For all $a, b \in E$, $a \wedge b = 0$ implies $a \perp b$.*
(c) *For all $a, b \in E$, $a \ominus (a \wedge b) \perp b$.*
(d) *For all $a, b \in E$, there exist $a_1, b_1, c \in E$ such that $a_1 \oplus b_1 \oplus c$ exists, $a_1 \oplus c = a$ and $b_1 \oplus c = b$.*

Let D be a bounded distributive lattice. Up to isomorphism, there exists a unique Boolean algebra $B(D)$ such that D is a $0,1$-sublattice of $B(D)$ and B generates $B(D)$ as a Boolean algebra. This Boolean algebra is called R-generated Boolean algebra. We refer to [7], section II.4, for an overview of results concerning R-generated Boolean algebras. See also [9] and [11]. For every element x of $B(D)$, there exists a finite chain $x_1 \leq \ldots \leq x_n$ in D such that $x = x_1 + \ldots + x_n$. Here, $+$ denotes the symmetric difference, as in Boolean rings. We then say than $\{x_i\}_{i=1}^n$ is a *D-chain representation* of x. It is easy to see that every element of $B(D)$ has a D-chain representation of even length.

Theorem 2. (Main result). *Let M be an MV-effect algebra. The mapping $\phi_M : B(M) \to M$ given by*

$$\phi_M(x) = \bigoplus_{i=1}^{n} (x_{2i} \ominus x_{2i-1}),$$

where $\{x_i\}_{i=1}^{2n}$ is a M-chain representation of x, is a surjective homomorphism of effect algebras.

2. $B(M)$

Let L be a lattice. An element a of L is *join-irreducible* iff $a = b \vee c$ implies that $a = b$ or $a = c$; it is *meet-irreducible* iff $a = b \wedge c$ implies that $a = b$ or $a = c$. The set of all nonzero join-irreducible elements of a lattice L is denoted by $J(L)$, the set of all non-unit meet-irreducible elements of a lattice L is denoted by $M(L)$.

Let L be a finite distributive lattice. Then the mapping $r : L \to 2^{J(L)}$ given by $r(x) = \{a \in J(L) : a \leq x\}$ is a $0,1$-embedding of L into $2^{J(L)}$. It is easy to check that $a \in J(L)$ iff $\{x \in L : x \not\geq a\}$ is a prime ideal and then $m(a) = \bigvee\{x \in L : x \not\geq a\} \in M(L)$. By a dual argument, it is easy to see that $a \mapsto m(a)$ is a bijection from $J(L)$ onto $M(L)$. Moreover, for every maximal chain C of L the mapping π_C given by $\pi_C(a) = \bigwedge\{x \in C : x \geq a\}$ is a bijection from the set of all join-irreducible elements onto C (see [7], Corollary II.1.14). Note that π_C maps nonzero elements onto nonzero elements.

In what follows, $\succ$ denotes the usual covering relation on a poset, that means, i$a \succ b$ iff b is a maximal element of the set $\{x : x < a\}$.

Lemma 3. *Let L be a finite distributive lattice, let C be a maximal chain in L, let $a \in J(L)$. Let $x \in C$, $\pi_C(a) \succ x$. Then $a \vee x = \pi_C(a)$ and $a \wedge x = a \wedge m(a)$.*

Proof. We have $\pi_C(a) \wedge (a \vee x) = (\pi_C(a) \wedge a) \vee (\pi_C(a) \wedge x) = a \vee x$, so $\pi_C(a) \geq a \vee x \geq x$. Since $\pi_C(a) \succ x$, we have either $\pi_C(a) = a \vee x$ or $a \vee x = x$. However, $a \vee x = x$ contradicts with $\pi_C(a) \neq x$, hence $\pi_C(a) = a \vee x$.

Since $x \not\geq a$, we have $x \leq m(a)$ and $a \wedge x \leq a \wedge m(a) \leq a$. Since $a \vee x = \pi_C(a) \succ x$, $a \succ a \wedge x$. Therefore, $a \wedge x = a \wedge m(a)$ or $a \wedge m(a) = a$. Since $a \not\leq m(a)$, $a \wedge x = a \wedge m(a)$. □

Corollary 4. *Let L be a finite sublattice of an MV-effect algebra M. Let C be a maximal chain of L, let $a \in J(L)$. Let $x \in C$, $\pi_C(a) \succ_L x$. Then $\pi_C(a) \ominus x = a \ominus (a \wedge m(a))$.*

Proof. Since M is a distributive lattice, L is distributive. By Lemma 3, we have $a \vee x = \pi_C(a)$ and $a \wedge x = a \wedge m(a)$. This implies that $\pi_C(a) \ominus x = (a \vee x) \ominus x = a \ominus (a \wedge x) = a \ominus (a \wedge m(a))$. □

Corollary 5. *Let L be a finite sublattice of an MV-effect algebra M. Let C_1, C_2 be maximal chains of L. There exists a bijection $b : C_1 \to C_2$ such that, for all $x_1, x_2 \in C_1$ with $x_2 \succ_L x_1$, $x_2 \ominus x_1 = b(x_2) \ominus y$, where $y \in C_2$ and $b(x_2) \succ_L y$.*

Proof. Since M is distributive, L is distributive. Let us put $b(x) = \pi_{C_2}(\pi_{C_1}^{-1}(x))$. Obviously, b is a bijection. Write $a = \pi_{C_1}^{-1}(x_2)$. By Corollary 4, $\pi_{C_1}(a) \ominus x_1 = x_2 \ominus x_1 = a \ominus a \wedge m(a)$. Similarly, by Corollary 4, $b(x_2) \ominus y = \pi_{C_2}(a) \ominus y = a \ominus a \wedge m(a)$. Thus, $x_2 \ominus x_1 = b(x_2) \ominus y$. □

Lemma 6. *Let L be a finite $0, 1$-sublattice of an MV-effect algebra M. Then the mapping $\psi_L : 2^{J(L)} \to M$ given by*

$$\psi_L(X) = \bigoplus_{a \in X} a \ominus (a \wedge m(a))$$

is a homomorphism of effect algebras and, for all $x \in L$, $\psi_L(r(x)) = x$.

Proof. By definition, $\psi_L(\emptyset) = 0$. Let $x \in L$ and write $L_x = \{y \in L : y \leq x\}$. Note that $r(x) = J(L_x)$. Let $C = \{0 = x_0, x_1, \ldots, x_n = x\}$ with $x_{i+1} \succ_L x_i$ be a maximal chain of L_x. Then the sum

$$\bigoplus_{i=1}^{n} x_i \ominus x_{i-1}$$

exists in M and equals x. By Corollary 4,

$$x_i \ominus x_{i-1} = \pi_C^{-1}(x_i) \ominus (\pi_C^{-1}(x_i) \wedge m(\pi_C^{-1}(x_i)).$$

Since π_C is a bijection, we have $r(x) = \{\pi_C^{-1}(x_i) : i \in \{1, \ldots, n\}\}$, hence $\psi_L(r(x))$ exists and equals x. As a consequence, $\psi_L(2^{J(L)}) = \psi_L(r(1)) = 1$. The additivity of ψ_L is trivial. □

Since, for every finite distributive lattice L, $r(L)$ R-generates $2^{J(L)}$, the injective mapping $r : L \to 2^{J(L)}$ uniquely extends to an isomorphism of Boolean algebras $\hat{r} : B(L) \to 2^{J(L)}$.

Lemma 7. *Let L be a finite $0,1$-sublattice of an MV-effect algebra M. Let $\psi_L, \hat{r}$ be the mappings given above. Then $\psi_L \circ \hat{r}$ is a homomorphism of effect algebras satisfying*

$$\psi_L \circ \hat{r}(x_1 + \ldots + x_{2n}) = \bigoplus_{i=1}^{n} (x_{2i} \ominus x_{2i-1})$$

for every chain $x_1 \leq \ldots \leq x_{2n}$ of L.

Proof. Evidently, $\psi_L \circ \hat{r} : B(L) \to M$ is a homomorphism of effect algebras. Let $x_1 \leq \ldots \leq x_{2n}$ be a chain in L. Then

$$\psi_L(\hat{r}(x_1 + \ldots + x_{2n})) = \psi_L(\hat{r}(x_1) + \ldots + \hat{r}(x_{2n})) = \psi_L(r(x_1) + \ldots + r(x_{2n})).$$

Since r is a lattice homomorphism, $r(x_1) \leq \ldots \leq r(x_{2n})$. Thus, in the Boolean algebra $2^{J(L)}$ we obtain

$$r(x_1) + \ldots + r(x_{2n}) = \bigoplus_{i=1}^{n} (r(x_{2i}) \ominus r(x_{2i-1})).$$

Finally, by Lemma 6,

$$\psi_L(r(x_1) + \ldots + r(x_{2n})) = \bigoplus_{i=1}^{n} \psi_L(r(x_{2i})) \ominus \psi_L(r(x_{2i-1})) = \bigoplus_{i=1}^{n} (x_{2i} \ominus x_{2i-1}) = \phi_L(x_1 + \ldots x_{2n}).$$

□

Proof of the main result. Let $x_1 \leq \ldots \leq x_{2n}$, $y_1 \leq \ldots \leq y_{2m}$ be two chains of M. Let L be the $0,1$-sublattice of M generated by $\{x_1, \ldots, x_{2n}, y_1, \ldots, y_{2m}\}$. Then $B(L)$ is a Boolean subalgebra of $B(M)$, $\{x_1, \ldots, x_{2n}, y_1, \ldots, y_{2m}\} \subseteq B(L)$ and, by Lemma 7, $\phi_L : B(L) \to M$ is a homomorphism of effect algebras.

Let us prove that ϕ_M is well defined. Suppose that $x_1 + \ldots + x_{2n} = y_1 + \ldots + y_{2m}$. By Lemma 7, $\bigoplus_{i=1}^{n}(x_{2i} \ominus x_{2i-1}) = \bigoplus_{i=1}^{n}(y_{2i} \ominus y_{2i-1})$, hence ϕ_M is well defined on $B(L)$ and hence on the whole set M. Moreover, ϕ_L is just the restriction of ϕ_M to $B(L)$.

Suppose now that $x = x_1 + \ldots + x_{2n} \perp y_1 + \ldots + y_{2m} = y$. Again, by Lemma 7, $\phi_L(x) \perp \phi_L(y)$ and $\phi_L(x \oplus y) = \phi_L(x) \oplus \phi_L(y)$. Obviously, $\phi_M(1) = 1$.

For the proof of surjectivity, it suffices to observe that, for all $x \in M$, $\phi_M(x) = x$. □

Example 8. Let M be MV-effect algebra, which is totally ordered. By [7], Corollary II.4.19, $B(M)$ is isomorphic to the Boolean algebra of all subsets of M of the form $[a_1, b_1)\dot{\cup} \ldots \dot{\cup}[a_n, b_n)$. Here, we denote $[a, b) = \{x \in M : a \leq x < b\}$. The ϕ_M is then given by

$$\phi_M([a_1, b_1)\dot{\cup} \ldots \dot{\cup}[a_n, b_n)) = (b_1 \ominus a_1) \oplus \ldots \oplus (b_n \ominus a_n).$$

Example 9. In this example, $[0, 1]$ denotes the closed real unit interval. Let $C_{[0,1]}$ be the MV-effect algebra of all real continuous functions $f : [0, 1] \to [0, 1]$. Let B be the Boolean algebra

$$\prod_{x \in [0,1]} B([0, 1]),$$

where $B([0,1])$ is the Boolean algebra of semiopen intervals described in Example 8. It is obvious that $C_{[0,1]}$, as a lattice, can be embedded into B by a mapping $\gamma : E \to B$ given by $\gamma(f) = ([f(x),0))_{x\in[0,1]}$. The image of E under γ then generates a Boolean subalgebra of B, which we can identify with $B(C_{[0,1]})$. The $\phi_{C_{[0,1]}} : B(C_{[0,1]}) \to C_{[0,1]}$ mapping can then be constructed as follows.

Let $(A_x)_{x\in[0,1]} \in B(C_{[0,1]})$. Fix $x \in [0,1]$ and write $A_x = [a_1,b_1)\dot\cup\ldots\dot\cup[a_n,b_n)$. The value of the continuous function $\phi_{C_{[0,1]}}((A_x)_{x\in[0,1]})$ at x is then equal to $(b_1 \ominus a_1) \oplus \ldots \oplus (b_n \ominus a_n)$.

An element x of an effect algebra E is called *central* iff $x \wedge x' = 0$ and every element $a \in E$ admits a decomposition $a = a_1 \oplus a_2$, where $a_1 \leq x$, $a_2 \leq x'$. There is a natural correspondence between complementary pairs of central elements and direct decompositions of E. The set of all central elements of E is denoted by $C(E)$. Central elements of effect algebras were introduced in [8]. In an MV-algebra M, we have $x \in C(M)$ iff $x \wedge x' = 0$.

Theorem 10. *Let M be an MV-effect algebra. Then $\phi_M^{-1}(x) = \{x\}$ iff $x \in C(M)$.*

Proof.

$\Longrightarrow$: Suppose $\phi_M^{-1}(x) = \{x\}$ and that $x \notin C(M)$. Then $x\wedge x' > 0$ and $x\vee x' < 1$. Let $L = \{0, x \wedge x', x, x', x \vee x', 1\}$. Then L is a $0,1$-sublattice of M and we have $J(L) = \{1, x, x', x \wedge x'\}$, $M(L) = \{x \vee x', x', x, 0\}$. The m mapping for L is given by the following table.

a	1	x	x'	$x\wedge x'$
$m(a)$	$x \vee x'$	x'	x	0

We have $r(x) = \{x, x \wedge x'\}$. Consider the set $Y = \{1, x\} \subseteq J(L)$. We have

$$\begin{aligned}\psi_L(Y) =&(1 \ominus 1 \wedge m(1)) \oplus (x \ominus x \wedge m(x)) = (1 \ominus (x \vee x')) \oplus (x \ominus (x \wedge x')) = \\ &(x \wedge x') \oplus (x \ominus (x \wedge x')) = x.\end{aligned}$$

Thus, $Y \in \psi_L^{-1}(x)$. Since $\phi_M^{-1}(x) = \phi_L^{-1}(x) = \{x\}$, $\psi_L^{-1}(x) = \{r(x)\}$. This implies that $Y = r(x)$ and $x \wedge x' = 1$, which is impossible.

$\Longleftarrow$: Suppose that $x \in C(M)$ and that there exists $y \in B(M)$ such that $x \neq y$ and $\phi_M(y) = x$. There exists a finite $0,1$-sublattice L of M such that $x, x', y \in B(L)$ and, since $2^{J(L)}$ and $B(L)$ are isomorphic, the set $Y = \hat{r}^{-1}(y) \subseteq J(L)$ satisfies $\psi_L(Y) = x$ and $Y \neq r(x)$.

Suppose that there exists $a \in Y$, $a \notin r(x)$. Since $x \in C(M)$, we have $x \wedge x' = 0$ and $x \vee x' = 1$; hence $r(x) \cap r(x') = \emptyset$ and $r(x) \cup r(x') = J(L)$. Therefore, $a \notin r(x)$ implies that $a \in r(x')$ and $a \leq x'$, so $a \ominus (a \wedge m(a)) \leq x'$. Since $a \in Y$, $a \ominus (a \wedge m(a)) \leq \psi_L(Y) = x$. This implies that $a \ominus (a \wedge m(a)) \leq x \wedge x' = 0$ and we obtain $a = a \wedge m(a)$. This is a contradiction.

Suppose that there exists $a \notin Y$, $a \in r(x)$. This implies that $a \notin r(x')$, $a \in J(L) \setminus Y$ and we have $\psi_L(J(L) \setminus Y) = x'$. By above paragraph, this leads to a contradiction. □

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