ON THE GENERALIZED VON KÁRMÁN SYSTEM FOR VISCOELASTIC PLATES. II. LONG MEMORY MODEL

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Abstract

We deal with the system of quasistationary von Kármán equations describing moderately large deflections of thin viscoelastic plates. We shall concentrate on a long memory material, which gives rise to a quasistationary system with a linear integro-differential main part and a nonlinear integro-differential term. The existence and the uniqueness of a solution as the limit of a semidiscrete approximation is verified. Its behaviour for large values of a time variable is studied.

Key words: von Kármán equations, viscoelastic plate, integro-differential equation, semidiscretization, existence, uniqueness

MSC (2000): 74D10, 74K20, 45K05, 65R20

1 Introduction

We continue the investigation of the behaviour of a viscoelastic isotropic plate involving the geometrical nonlinearity. The behaviour of a solution corresponds to moderately large deflections due to the theory of Fox, Raoult and Simo [3]. We assume the bounded middle surface Ω of the plate with a Lipschitz continuous boundary Γ. We have formulated in the first part of the paper [1] the integro-differential von Kármán system for the deflection $w(t,x)$ and the Airy stress function $\Phi(t,x),\ t \geq 0,\ x \in \Omega:$

$$D(0)\Delta^2 w + D' * \Delta^2 w - [\Phi, w] = f(t),$$
$$\Delta^2 \Phi = -\frac{h}{2}(E(0)[w,w] + E' * [w,w]),$$

where $E \in C^1(R^+)$ is a positive decreasing relaxation function, $t \to D(t) = \frac{h^3}{12(1-\mu^2)}E(t)$ is the material function, $h > 0$ the thickness of a plate, $\mu \in (0, 1/2)$ the Poisson ratio, $(f * g)(t) = \int_0^t f(t-s)g(s)ds$ the convolution product and

$$[v, w] = \partial_{11}v\partial_{22}w + \partial_{22}v\partial_{11}w - 2\partial_{12}v\partial_{12}w, \quad v,\ w \in H^2(\Omega).$$

In the special case of the relaxation function

$$E(t) = E_0 + E_1 e^{-\beta t},\ t \geq 0,\ E_0 > 0,\ E_1 > 0,\ \beta > 0$$

the original integro-differential system is in [1] transformed into the nonlinear pseudoparabolic system. We considered the clamped plate with Dirichlet boundary conditions for both the deflection and the Airy stress function. The initial-boundary value for the original system was expressed as the nonlinear initial value problem for the pseudoparabolic equation in the Sobolev space $H_0^2(\Omega)$. We have substituted this problem by a finite sequence of stationary von Kármán-like equations for every time step. The corresponding sequence of segment line functions was convergent to a nonstationary deflection function.

We shall deal here with the general long memory isotropic case, considering mixed boundary conditions for the deflection and the nonhomogeneous conditions for the Airy stress function formulated in a similar way as in [6] or [11] for the elastic plate.

The existence of a weak solution of the resulting nonlinear integro-differential system will be verified as the limit of the sequences of segment line and step in time functions after substituting the convolution integrals by finite sums.

The main condition for the convergence is the estimate of the right-hand side, which do not depend on the lenght of the time interval.

We consider the memory term also in the equation for the Airy stress function. The dynamic viscoelastic von Kármán systems are studied nowadays mainly in the framework of controllability problems. The authors (Horn and Lasiecka [7], Lagnese [9], Muñoz Rivera and Perla Menzala [10]) have considered the memory term only in the linear part of the system.

The last chapter of the article is devoted to the study of the behaviour of a solution for large values of time variable. We shall verify that only the estimate of the limit value of the right-hand side implies the limit behaviour of the solution to the corresponding solution of the stationary von Kármán system.

2 Formulation of the problem

We assume that a plate is subjected both to a perpendicular load of a plane density f and the forces acting along the boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, where each Γ_i is either empty or $mes(\Gamma_i) > 0$. Further we assume that $\Gamma_1 \neq \emptyset$ or $\Gamma_2 \neq \emptyset$ and Γ_2 is not a segment of a straight line. The part Γ_3 contains only smooth parts.

We shall consider the following boundary value problem:

$$D(0)\Delta^2 w + D' * \Delta^2 w - [\Phi, w] = f(t), \tag{1}$$

$$w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_1, \tag{2}$$

$$w = 0, \quad \mathcal{M}(w) + k_2 \frac{\partial w}{\partial \nu} = m_2 \text{ on } \Gamma_2, \tag{3}$$

$$\mathcal{M}(w) + k_{31} \frac{\partial w}{\partial \nu} = m_3, \quad \mathcal{S}(w) + k_{32} w = t_3 \text{ on } \Gamma_3, \tag{4}$$

$$\Delta^2 \Phi - \frac{h}{2}(E(0)[w, w] + E' * [w, w]), \tag{5}$$

$$\Phi = \phi_0, \quad \frac{\partial \Phi}{\partial \nu} = \phi_1 \quad \text{on } \Gamma, \tag{6}$$

where

$$\begin{aligned}
\mathcal{M}(w) &= D(0)M(w) + D' * M(w), \\
M(w) &= \mu \Delta w + (1-\mu)(w_{,11}\nu_1^2 + 2w_{,12}\nu_1\nu_2 + w_{,22}\nu_2^2), \\
\mathcal{S}(w) &= w_{,1}\Phi_{,2\sigma} - w_{,2}\Phi_{,1\sigma} + D(0)S(w) + D' * S(w), \\
S(w) &= -\frac{\partial}{\partial \nu}\Delta w + (1-\mu)\frac{\partial}{\partial \sigma}[w_{,11}\nu_1\nu_2 - w_{,12}(\nu_1^2 - \nu_2^2) - w_{,22}\nu_1\nu_2].
\end{aligned}$$

We set

$$w_{,i} = \frac{\partial w}{\partial x_i}, \; w_{,ij} = \frac{\partial^2 w}{\partial x_i \partial x_j}, \; \Phi_{,i\sigma} = \frac{\partial}{\partial \sigma}\frac{\partial \Phi}{\partial x_i}.$$

$\nu = (\nu_1, \nu_2), \quad \sigma = (-\nu_2, \nu_1)$ are the unit outward normal and the unit tangential vector with respect to Γ respectively.

The functions $k_2 \geq 0, \; k_{3i} \geq 0, \; i = 1, 2$ satisfy the conditions

$$k_2 \in L^p(\Gamma_2), \; k_{31} \in L^p(\Gamma_3), \; p > 1, \; k_{32} \in L^1(\Gamma_3).$$

They express the elastic contact of the boundary in the case of their positiveness.

Let us introduce following Hilbert spaces corresponding to the boundary conditions (2)-(4) and (6). We set

$$H_0^2(\Omega) = \{v \in H^2(\Omega)|\ v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma\}.$$

$H_0^2(\Omega)$ is the Hilbert space with the inner product $((.,.))_0$ and the norm $\|.\|_0$ defined by

$$((u,v))_0 = \int_\Omega \Delta u \Delta v dx, \quad \|u\|_0 = ((u,u))_0^{1/2}, \quad u,v \in H_0^2(\Omega).$$

Further we introduce the Hilbert space

$$V = \{v \in H^2(\Omega)|\ v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_1, \quad v = 0 \text{ on } \Gamma_2\}$$

with the inner product $((.,.))$ and the norm $\|.\|$ defined by

$$\begin{aligned} &((u,v)) = \\ &\int_\Omega [u_{,11}v_{,11} + 2(1-\mu)u_{,12}v_{,12} + u_{,22}v_{,22} + \mu(u_{,11}v_{,22} + u_{,22}v_{,11}]dx \\ &+ \int_{\Gamma_2} k_2 \frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu} d\sigma + \int_{\Gamma_3} (k_{31}\frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu} + k_{32}uv) d\sigma, \quad (7) \\ &\|u\| = ((u,u))^{1/2}, \quad u,v \in V. \quad (8) \end{aligned}$$

The norm defined in (8) is in the space V equivalent with the original norm

$$\|u\|_{H^2(\Omega)} = [\int_\Omega (u^2 + u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)dx]^{1/2}$$

of the Sobolev space $H^2(\Omega)$ (see [11], Lemma 11.3.2 for the details).

We denote by V^* the space of all linear bounded functionals over V with the norm $\|f\|_*$ and the duality pairing $\langle f, v\rangle$ for $f \in V^*$ and $v \in V$.

Finally we impose the conditions upon the right-hand sides in the Problem (1)-(6). We assume

$$f \in W^{1,2}(0,T;V^*), \quad (9)$$

$$m_i \in W^{1,2}(0,T;L^p(\Gamma_i)), \quad i = 2,\ 3 \quad (10)$$

$$t_3 \in W^{1,2}(0,T;L^p(\Gamma_3)), \quad T > 0. \quad (11)$$

For any Banach space X we denote by $W^{1,2}(0,T;X)$ the space of functions $f \in L^2(0,T;X)$ such that $f' \in L^2(0,T;X)$, where f' is the derivative in the sense of distributions $\mathcal{D}'(0,T;X)$ of the function f. It can be verified in the same way as for real functions that $W^{1,2}(0,T;X)$ is a Banach space with a norm $\|f\|_{W^{1,2}} = \|f\|_{L^2(0,T;X)} + \|f'\|_{L^2(0,T;X)}$. (See [2] for further properties of the space $W^{1,2}(0,T;X)$).

We denote by $C(0,T;X)$ the Banach space of continuous functions defined on the interval $[0,T]$ with values in X.

We suppose the functions $\phi_i : [0,T] \times \Gamma \to R,\ i = 0,1$ to be sufficiently smooth in order to enable the existence of a function

$$F \in W^{1,2}(0,T;H^2(\Omega))$$

such that

$$F = \phi_0, \quad \frac{\partial F}{\partial \nu} = \phi_1 \quad \text{on } \Gamma, \tag{12}$$

$$((F(t),\phi))_0 = 0 \quad \text{for all } \phi \in H_0^2(\Omega). \tag{13}$$

The paper [6] contains the detailed assumptions imposed upon $\phi_0,\ \phi_1$ in order to fulfil (12), (13). For every $t \in [0,T]$ is $F(t) \in H^2(\Omega)$ a weak solution of the biharmonic equation $\Delta^2 F(t) = 0$ with the boundary conditions (12).

Let us introduce the trilinear form

$$\begin{aligned}
&\mathcal{B}(u,v;w) = \int_\Omega [(u_{,12}v_{,2} - u_{,22}v_{,1})w_{,1} + (u_{,12}v_{,1} - u_{,11}v_{,2})w_{,2}]dx, \\
&u,v,w \in H^2(\Omega).
\end{aligned} \tag{14}$$

The existence of the integral in (14) is assured due to the imbedding $H^2(\Omega) \subset W^{1,4}(\Omega)$. The form $\mathcal{B}$ fulfils the inequality

$$|\mathcal{B}(u,v;w)| \le \sqrt{2}|u|_{H^2(\Omega)}|v|_{W^{1,4}(\Omega)}|w|_{W^{1,4}(\Omega)}, \quad u,v,w \in H^2(\Omega) \tag{15}$$

with seminorms

$$\begin{aligned}
|u|_{H^2(\Omega)} &= [\int_\Omega (u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)dx]^{1/2}, \\
|v|_{W^{1,4}(\Omega)} &= [\int_\Omega (v_{,1}^4 + v_{,2}^4)dx]^{1/4}.
\end{aligned}$$

After multiplying the equations (1) and (5) with test functions $v \in V$ and $\phi \in H_0^2(\Omega)$ respectively and applying the boundary conditions we arrive to a formulation of a weak solution of the problem (1)-(6) in a similar way as in [6] for the elastic case.

Definition 2.1 *A pair* $\{w, \Phi\}$ *is a weak solution of the boundary value problem (1)-(6) if*

1. $w \in C([0,T], V)$,
2. $\Phi \in C([0,T], H^2(\Omega)), \quad \Phi = \phi_0, \quad \frac{\partial \Phi}{\partial \nu} = \phi_1 \quad on\ \Gamma,$
3. *There hold the identities*

$$((D(0)w(t) + D' * w(t), v)) - \mathcal{B}(\Phi(t), w(t); v) =$$
$$\int_{\Gamma_2} m_2(t)\frac{\partial v}{\partial \nu} d\sigma + \int_{\Gamma_3} (m_3(t)\frac{\partial v}{\partial \nu} + t_3(t)v) d\sigma + \langle f(t), v\rangle, \tag{16}$$

for all $v \in V$,

$$((\Phi(t), \phi))_0 = -\frac{h}{2}\int_\Omega (E(0)[w,w] + E' * [w,w])(t)\phi dx \tag{17}$$

for all $v \in H_0^2(\Omega)$.

After expressing the Airy stress function Φ in the form $\Phi = F + \Psi$, where a function F is defined in (12), (13) we can directly derive the following

Theorem 2.2 *A pair* $\{w, \Phi\}$ *is a weak solution of the boundary value problem (1)-(6) if and only if* $\Phi = \Psi + F$ *and a pair* $\{w, \Psi\} \in C([0,T], V) \times C([0,T], H_0^2(\Omega))$ *satisfies the identities*

$$((D(0)w(t) + (D' * w)(t), v)) - \mathcal{B}(\Psi(t) + F(t), w(t); v) =$$
$$\int_{\Gamma_2} m_2(t)\frac{\partial v}{\partial \nu} d\sigma + \int_{\Gamma_3} (m_3(t)\frac{\partial v}{\partial \nu} + t_3(t)v) d\sigma + \langle f(t), v\rangle, \tag{18}$$

for all $v \in V$,

$$((\Psi(t), \phi))_0 = -\frac{h}{2}\int_\Omega (E(0)[w,w](t) + (E' * [w,w])(t))\phi dx \tag{19}$$

for all $\phi \in H_0^2(\Omega)$.

Before transforming the system (18), (19) into one canonical Volterra type nonlinear integral equation in the Hilbert space V we derive some properties of the trilinear form $\mathcal{B}$. We shall use a well known formula ([5])

$$\int_\Omega [u,v]\phi dx = \int_\Omega u[v,\phi] dx \quad \text{for all } u,\ v \in V,\ \phi \in H_0^2(\Omega).$$

Applying the integration by parts and the density of the sets $C_0^\infty(\Omega)$ and $C^\infty(\overline{\Omega})$ in $H_0^2(\Omega)$ and $H^2(\Omega)$ respectively we arrive at the formula

$$\mathcal{B}(u, w; v) = \mathcal{B}(v, w; u) = \mathcal{B}(w, v; u) \quad \text{for all } u,\ w \in H^2(\Omega),\ v \in H_0^2(\Omega). \tag{20}$$

The following symmetry property

$$\mathcal{B}(u,v;w) = \mathcal{B}(u,w;v) \quad \text{for all } u,\ v,\ w \in H^2(\Omega) \tag{21}$$

can be seen directly. Using the inequality (15) we obtain the inequalities

$$|\mathcal{B}(F,u;v)| \le c_1 \|F\|_{H^2(\Omega)} \|u\|_{W^{1,4}(\Omega)} \|v\| \tag{22}$$

for all $F \in H^2(\Omega),\ u,\ v \in V$,

$$|\mathcal{B}(\phi,u;v)| \le c_2 \|\phi\|_0 \|u\|_{W^{1,4}(\Omega)} \|v\| \tag{23}$$

for all $\phi \in H_0^2(\Omega),\ u,\ v \in V$,

We introduce the bilinear operators $B : H^2(\Omega) \times H^2(\Omega) \to V$ and $B_0 : V \times V \to H_0^2(\Omega)$ as solutions of equations

$$((B(u,w),v)) = \mathcal{B}(u,w;v) \quad \text{for all } v \in V, \tag{24}$$

$$((B_0(u,w),\phi))_0 = \int_\Omega [u,w]\phi dx \quad \text{for all } \phi \in H_0^2(\Omega). \tag{25}$$

Both equations are solved uniquelly, because the right-hand sides of both relations belong to the dual spaces V^* and $(H_0^2(\Omega))^*$ respectively.

The operators $B : H^2(\Omega) \times H^2(\Omega) \to V,\ B_0 : V \times V \to H_0^2(\Omega)$ are bounded (as bilinear operators) and satisfy the relations

$$\int_\Omega [u,v]\phi dx = ((B(u,v),\phi)) = ((B(v,u),\phi)) = ((B(u,\phi),v)) \tag{26}$$

for all $u,v \in V,\ \phi \in H_0^2(\Omega)$,

$$B_0(u,v) = B_0(v,u) \quad \text{for all } u,v \in V, \tag{27}$$

$$((B(B_0(u,v),w),\phi)) = ((\ B_0(u,v), B_0(w,\phi)\ ))_0 \tag{28}$$

for all $u,v,w,\phi \in V$,

$$\|B(u,v)\| \le c_3 \|u\|\ \|w\|_{W^{1,4}(\Omega)}, \tag{29}$$

$$\|B(u,v)\| \le \|B\|\ \|u\|\ \|v\| \tag{30}$$

for all $u,v \in H^2(\Omega)$,

$$\|B_0(u,v)\|_0 \le c_4 \|u\|_{W^{1,4}(\Omega)} \|w\|_{W^{1,4}(\Omega)} \tag{31}$$

$$\|B_0(u,v)\|_0 \le \|B_0\|\ \|u\|\ \|v\|, \tag{32}$$

for all $u,v \in V$.

Applying the operator B_0 we express the function Ψ from the identity (19) in the form

$$\Psi(t) = -\frac{h}{2}[E(0)B_0(w,w)(t) + E' * B_0(w,w)(t)],\ t \in [0,T]. \tag{33}$$

Let us define the function $q : [0,T] \to V$ by the relation

$$((q(t),v)) = \\ \frac{1}{D(0)}[\int_{\Gamma_2} m_2(t)\frac{\partial v}{\partial \nu}d\sigma + \int_{\Gamma_3}(m_3(t)\frac{\partial v}{\partial \nu} + t_3(t)v)d\sigma + \langle f(t),v\rangle] \quad (34)$$

for all $v \in V$.

The elements $q(t) \in V$ are uniquelly defined as the Riesz representants of the right-hand side in the relation (34) which is for every $t \in [0,T]$ the linear continuous functional over V. Moreover we have the regularity

$$q \in W^{1,2}(0,T;V) \quad (35)$$

due to the assumptions (9)-(11).

After inserting the values $\Psi(t)$ from (33) into (18) and using the relations (24), (34) we arrive at

The canonical Volterra integral equation

$$w(t) + g * w - aB(F(t),w(t)) + \\ \alpha B(\ B_0(w,w)(t) + g * B_0(w,w)(t), w(t)\ ) = q(t) \in V, \quad (36) \\ g(t) = \frac{D'(t)}{D(0)} = \frac{E'(t)}{E(0)}, \quad a = \frac{1}{D(0)}, \quad \alpha = \frac{h}{2}\frac{E(0)}{D(0)}.$$

It can be readily seen that a function w is a solution of the canonical equation (36) if and only if a pair $\{w, B_0(w,w)\} : [0,T] \to V \times H_0^2(\Omega)$ is a solution of the identities (18), (19) and hence a pair $\{w, F + B_0(w,w)\}$ is a weak solution of the original problem (1)-(6).

3 Existence and Uniqueness of a Solution

We shall verify the existence of a solution of the canonical integro-differential equation (36) using its discretization with respect to the time variable t.

Before formulating the discrete scheme let us set some additional growth assumptions on the kernel function g and the bounds on the function F. We assume the exponential behaviour of the continuous kernel function g:

$$0 < -g(t) \le Ke^{-\beta t},\ t \ge 0,\ 0 < K < \beta. \quad (37)$$

corresponding to the most of viscoelastic materials (see [4] for example). Further we assume that

$$((B(F(t),v),v)) \le 0\ \forall\ t \in [0,T],\ v \in V. \quad (38)$$

Comparing with (14), (24) we can see that the condition

$$\int_\Omega [F_{,22}(t)(v_{,1})^2 - 2F_{,12}(t)v_{,1}v_{,2} + F_{,11}(t)(v_{,2})^2]dx \geq 0 \,\forall\, t \in [0,T],\; v \in V \quad (39)$$

is sufficient for fulfilling (38).

For a fixed integer N we set

$$\tau = \tfrac{T}{N},\; t_i = i\tau,\; w_i = w(t_i),\; i = 0,1,...,N;$$
$$\delta w_j = \tfrac{1}{\tau}(w_j - w_{j-1}),\; j = 1,...,N.$$

We convert nonlinear Volterra integral equation (36) into a finite sequence of nonlinear stationary equations in the Hilbert space V which are similar to the canonical von Kármán equations for the elastic plate. We shall use the Rothe's method in a similar way as by Kačur [8] or Slodička [12] in the case of parabolic integro-differential equations. Applying the discrete values w_i instead of $w(t_i),\; i = 0,1,...,N$ and the substituting integrals in (36) by finite sums we arrive at the equations

$$w_0 - aB(F_0, w_0) + \alpha B(B_0(w_0, w_0), w_0) = q_0, \quad (40)$$

$$w_i - aB(F_i, w_i) + \tau \sum_{j=0}^{i-1} g_{i-j} w_j +$$
$$\alpha B\left(B_0(w_i, w_i) + \tau \sum_{j=0}^{i-1} g_{i-j} B_0(w_j, w_i)\right) = q_i, \quad (41)$$
$$i = 1,...,N.$$

The equations (40), (41) are the Euler equations for the functionals

$$J_0(v) = \frac{1}{2}[\|v\|^2 - a((B(F_0,v),v))] + \frac{a}{4}\|B_0(v,v)\|_0^2$$
$$J_i(v) = \frac{1}{2}[\|v\|^2 - a((B(F_i,v),v))] + \frac{a}{4}\|B_0(v,v)\|_0^2 +$$
$$\left(\left(\tau \sum_{j=0}^{i-1} g_{i-j} w_j, v\right)\right) + \frac{\alpha}{2}\left(\left(\tau \sum_{j=0}^{i-1} g_{i-j} B_0(w_j, w_j), B_0(v,v)\right)\right)_0$$
$$-((q_i, v)),\; v \in V,\; i = 1,...,n.$$

The functionals $J_i,\; i = 0,1,...,N$ are weakly lower semicontinuous and coercive over V. The coerciveness

$$\lim_{\|v\| \to +\infty} J_i(v) = +\infty \quad (42)$$

can be seen directly. The weakly lower semicontinuity is the consequence of the inequality (31) and the compact imbedding $V \subset W^{1,4}(\Omega)$ which imply

$$v_n \rightharpoonup v \text{ in } V \Longrightarrow B_0(v_n, v_n) \to B_0(v,v) \text{ in } H_0^2(\Omega). \tag{43}$$

Then there exist elements $w_i \in V$ fulfilling the minimum condition

$$J_i(w_i) = \min_{v \in V} J_i(v), \ \ i = 0, 1, ..., N$$

and solving the discrete canonical equations (40), (41).

We proceed with a priori estimates. In order to achieve the best possible uniform estimates we multiply the discrete canonical equations (41) with the exponential functions with positive exponents. Let a constant $\gamma \in R$ fulfil the condition

$$0 < \gamma < \beta - K. \tag{44}$$

We start with estimates of the finite sums.

Lemma 3.1 *Let*

$$\omega_j = \|w_j\|^2 + \alpha \|B_0(w_j, w_j)\|_0^2, \ \ j = 0, 1, ..., N. \tag{45}$$

Then

$$\sum_{j=0}^{i} \tau e^{\gamma j \tau} \omega_j \le C_1(\beta, \gamma, K) \sum_{j=0}^{i} \tau e^{\gamma j \tau} \|q_j\|^2, \tag{46}$$

$$C_1(\beta, \gamma, K) = \frac{[\beta(\beta-\gamma)]^{3/2}}{K[\sqrt{\beta(\beta-\gamma)} - K]^2}.$$

Proof. We set $i = j$ in (41), multiply it with $\tau e^{\gamma \tau j} w_j$ in V and add for $j = 0, 1, ..., i$. After applying the property (38) we obtain subsequently the inequalities

$$\sum_{j=0}^{i} \tau e^{\gamma j \tau} \omega_j \le \sum_{j=1}^{i} \tau^2 e^{\gamma j \tau} \left(\left(\sum_{k=0}^{j-1} g_{j-k} w_k, w_j \right) \right)$$

$$+ \alpha \sum_{j=1}^{i} \tau^2 e^{\gamma j \tau} \left(\left(\sum_{k=0}^{j-1} g_{j-k} B_0(w_k, w_k), B_0(w_j, w_j) \right) \right)_0$$

$$+ \sum_{j=0}^{i} \tau e^{\gamma j \tau} ((q_j, w_j)),$$

$$
\sum_{j=0}^{i} \tau e^{\gamma j \tau} \omega_j \leq
$$

$$
\frac{\tau^3}{1-\epsilon} \sum_{j=1}^{i} e^{\gamma j \tau} \left[\| \sum_{k=0}^{j-1} g_{j-k} w_k \|^2 + \alpha \| \sum_{k=0}^{j-1} g_{j-k} B_0(w_k, w_k) \|_0^2 \right]
$$

$$
+ \frac{\tau}{\epsilon(1-\epsilon)} \sum_{j=0}^{i} e^{\gamma j \tau} \|q_j\|^2, \ 0 < \epsilon < 1.
$$

Using the growth assumption (37) and the convexity of the function $\|.\|^2$ we obtain the estimates

$$
\sum_{j=0}^{i} \tau e^{\gamma j \tau} \omega_j \leq \tag{47}
$$

$$
\frac{K^2}{1-\epsilon} \tau^3 \sum_{j=1}^{i} e^{(\gamma - 2\beta) j \tau} \sum_{k=0}^{j-1} e^{\beta k \tau} \sum_{k=0}^{j-1} e^{\beta k \tau} \omega_k + \frac{\tau}{\epsilon(1-\epsilon)} \sum_{j=0}^{i} e^{\gamma j \tau} \|q_j\|^2
$$

$$
= \frac{K^2}{1-\epsilon} \ \tau^3 \sum_{j=1}^{i} \frac{e^{-(\beta-\gamma) j \tau} - e^{-(2\beta - \gamma) j \tau}}{e^{\beta \tau} - 1} \sum_{k=0}^{j-1} e^{\beta k \tau} \omega_k
$$

$$
+ \frac{\tau}{\epsilon(1-\epsilon)} \sum_{j=0}^{i} e^{\gamma j \tau} \|q_j\|^2.
$$

We continue with the estimate of the double sum in the last inequality. We have

$$
\sum_{j=1}^{i} (e^{-(\beta-\gamma) j \tau} - e^{-(2\beta-\gamma) j \tau}) \sum_{k=0}^{j-1} e^{\beta k \tau} \omega_k \leq \sum_{j=1}^{i} \sum_{k=0}^{j-1} e^{\beta k \tau} e^{(\gamma - \beta) j \tau} \omega_k
$$

$$
= \sum_{k=0}^{i-1} e^{\beta k \tau} [\sum_{j=k+1}^{i} e^{(\gamma-\beta) j \tau}] \omega_k
$$

$$
= e^{-(\beta-\gamma)\tau} \sum_{k=0}^{i-1} e^{\gamma k \tau} \frac{1 - e^{(\gamma-\beta)(i-k)\tau}}{1 - e^{(\gamma-\beta)\tau}}
$$

$$
\leq \sum_{k=0}^{i-1} e^{\gamma k \tau} \frac{1}{e^{(\beta-\gamma)\tau} - 1} \omega_k \leq \frac{1}{(\beta-\gamma)\tau} \sum_{k=0}^{i-1} e^{\gamma k \tau} \omega_k.
$$

Comparing with (47) we obtain the inequality

$$\sum_{j=0}^{i} \tau e^{\gamma j \tau} \omega_j \leq \tag{48}$$
$$\frac{K^2}{(1-\epsilon)\beta(\beta-\gamma)} \sum_{j=0}^{i-1} \tau e^{\gamma j \tau} \omega_j + \frac{1}{\epsilon(1-\epsilon)} \sum_{j=0}^{i} \tau e^{\gamma j \tau} \|q_j\|^2.$$

Setting

$$\epsilon = 1 - \frac{K}{\sqrt{\beta(\beta-\gamma)}}$$

we obtain

$$\frac{K^2}{(1-\epsilon)\beta(\beta-\gamma)} = 1-\epsilon.$$

The inequality (48) then implies

$$\sum_{j=0}^{i} \tau e^{\gamma j \tau} \omega_j \leq \frac{1}{\epsilon^2(1-\epsilon)} \sum_{j=0}^{i} \tau e^{\gamma j \tau} \|q_j\|^2$$

and the estimate (46) folows after realizing that

$$\frac{1}{\epsilon^2(1-\epsilon)} = \frac{[\beta(\beta-\gamma)]^{3/2}}{K[\sqrt{\beta(\beta-\gamma)}-K]^2} = C_1(\beta,\gamma,K)$$

.

Remark 3.2 *It was possible to use a simpler approach in the obtaining the summation estimate. We could apply the discrete Gronwall lemma, but with significantly larger constant C_1, containing the length T of the time interval in (46).*

We continue with uniform a priori estimates.

Lemma 3.3 *There holds the estimate*

$$\|w_i\| \leq C_2(\beta,\gamma,K) \left[\sum_{j=0}^{i-1} \tau e^{-\gamma(i-j)\tau} \|q_j\|^2 \right]^{1/2} + \|q_i\|, \tag{49}$$
$$i = 1, 2, ..., N,$$
$$C_2(\beta,\gamma,K) = \frac{2\sqrt{2}K}{\sqrt{2\beta-\gamma}} C_1(\beta,\gamma,K)^{1/2}.$$

Proof. The equations (41), (45) imply the identity

$$\|w_i\|^2 + \alpha\|B_0(w_i, w_i)\|_0^2 = a((B(F_i, w_i), w_i)) + ((q_i, w_i))$$
$$-\left(\left(\tau\sum_{j=0}^{i-1} g_{i-j}w_j, w_i\right)\right) - \alpha\left(\left(\tau\sum_{j=0}^{i-1} g_{i-j}B_0(w_j, w_j), B_0(w_i, w_i)\right)\right)_0.$$

Employing the property (38) and Cauchy-Schwarz inequality we obtain the inequality

$$\|w_i\| \le \left\|\tau\sum_{j=0}^{i-1} g_{i-j}w_j\right\| + \sqrt{\alpha}\left\|\tau\sum_{j=0}^{i-1} g_{i-j}B_0(w_j, w_j)\right\|_0 + \|q_i\|. \qquad (50)$$

Again using the convexity of $\|.\|^2$ and the properties of exponential functions we arrive at the inequalities

$$e^{\gamma i\tau}\left\|\sum_{j=0}^{i-1} g_{i-j}w_j\right\|^2 = \left\|\sum_{j=0}^{i-1} g_{i-j}e^{\gamma(i-j)\tau/2}(e^{\gamma j\tau/2}w_j)\right\|^2$$
$$\le \sum_{j=0}^{i-1}(-g_{i-j}e^{\gamma(i-j)\tau/2})\ \sum_{j=0}^{i-1}(-g_{i-j}e^{\gamma(i-j)\tau/2})e^{\gamma j\tau}\|w_j\|^2 \qquad (51)$$
$$\le K^2\sum_{j=0}^{i-1} e^{-(\beta-\gamma/2)(i-j)\tau}\ \sum_{j=0}^{i-1} e^{-(\beta-\gamma/2)(i-j)\tau}e^{\gamma j\tau}\|w_j\|^2$$
$$\le K^2\frac{1-e^{-(\beta-\gamma/2)i\tau}}{e^{(\beta-\gamma/2)\tau}-1}\ \sum_{j=0}^{i-1} e^{\gamma j\tau}\|w_j\|^2 \le \frac{K^2}{(\beta-\gamma/2)\tau}\sum_{j=0}^{i-1} e^{\gamma j\tau}\|w_j\|^2.$$

In the same way we obtain

$$e^{\gamma i\tau}\left\|\sum_{j=0}^{i-1} g_{i-j}B_0(w_j, w_j)\right\|_0^2 \le \frac{K^2}{(\beta-\gamma/2)\tau}\sum_{j=0}^{i-1} e^{\gamma j\tau}\|B_0(w_j, w_j)\|_0^2. \qquad (52)$$

Combining (50), (51), (52) we obtain the inequality

$$\begin{aligned}\|w_i\| \ \le\ & \left(\frac{2K^2}{2\beta-\gamma}\right)^{1/2}\left(\tau\sum_{j=0}^{i-1} e^{-\gamma(i-j)\tau}\|w_j\|^2\right)^{1/2} \\ +\ & \left(\frac{2K^2\alpha}{2\beta-\gamma}\right)^{1/2}\left(\tau\sum_{j=0}^{i-1} e^{-\gamma(i-j)\tau}\|B_0(w_j, w_j)\|_0^2\right)^{1/2} + \|q_i\|\end{aligned}$$

and applying the estimate (46) we have

$$\|w_i\| \le \|q_i\| +$$
$$\frac{2\sqrt{2}K}{\sqrt{2\beta-\gamma}} C_1(\beta,\gamma,K)^{1/2} \left[\tau \sum_{j=0}^{i-1} e^{-\gamma(i-j)\tau} \|q_j\|^2\right]^{1/2}, \tag{53}$$
$$i = 1, ..., N.$$

The estimate (49) follows immediatly.

In order to achieve the convergence of the scheme we need the a priori estimate of the sum of differences δw_i. We impose the bounds assumption on the right-hand side q.

Lemma 3.4 *Let*

$$C_2(\beta,\gamma,K) \left[\sum_{j=0}^{i-1} \tau e^{-\gamma(i-j)\tau} \|q_j\|^2\right]^{1/2} + \|q_i\| \le \frac{1-\epsilon}{\sqrt{\alpha}\|B_0\|},\ \epsilon \in (0,1). \tag{54}$$

Then

$$\tau \sum_{j=1}^{N} \|\delta w_j\|^2 \le C_3(\beta,\gamma,K,\epsilon,T). \tag{55}$$

Proof. After setting $i = j,\ i = j-1$ in (41) and substracting we have the identities

$$\delta w_1 + g_1 w_0 + \delta B(\alpha B_0(w_1,w_1) - aF_1, w_1) + \alpha B(g_1 B_0(w_0,w_0), w_1) = \delta q_1,$$

$$\delta w_j + \delta B(\alpha B_0(w_j,w_j) - aF_j, w_j) + g_j w_0 + \tau \sum_{k=1}^{j-1} g_{j-k} \delta w_k +$$
$$\tau\alpha B(g_j B_0(w_0,w_0)_+ \sum_{k=1}^{j-1} g_{j-k} \delta B_0(w_k,w_k), w_j) = \delta q_j, \quad j = 2, ..., i.$$

After multiplying the last identities in the space V with $\tau\delta w_j,\ j = 1, ..., i$ and adding we arrive at

$$\tau \sum_{j=1}^{i} \|\delta w_j\|^2 - a\tau \sum_{j=1}^{i} ((B(\delta F_j, w_{j-1}) + B(F_j, \delta w_j)\ ,\ \delta w_j))$$

$$+\alpha\sum_{j=1}^{i}((B(B_0(w_j,w_j),w_j)-B(B_0(w_{j-1},w_{j-1}),w_{j-1})\ ,\ \delta w_j))$$

$$+\tau\sum_{j=1}^{i}((\ g_jw_0+\alpha g_jB(B_0(w_0,w_0),w_j),\delta w_j)\ )) \tag{56}$$

$$+\tau^2\sum_{j=2}^{i}((\sum_{k=1}^{j-1}g_{j-k}\delta w_k+\alpha B(g_{j-k}\delta B_0(w_k,w_k),w_j)\ ,\ \delta w_j))$$

$$=\tau\sum_{j=1}^{i}((\delta q_j,\delta w_j)).$$

Let us set

$$w_\xi=w_{j-1}+\xi(w_j-w_{j-1}),\ \xi\in R$$

for a fixed $j\in 1,...,i$. We have then the relation

$$((B(B_0(w_j,w_j),w_j)-B(B_0(w_{j-1},w_{j-1}),w_{j-1})\ ,\ \delta w_j))=$$

$$\tau\int_0^1[2\|B_0(\delta w_j,w_\xi)\|_0^2+((\ B_0(w_\xi,w_\xi),B_0(\delta w_j,\delta w_j)\ ))_0]d\xi. \tag{57}$$

Using the assumption (38) and the relation (57) we obtain from (56) the inequality

$$(1-\frac{\epsilon}{2})\tau\sum_{j=1}^{i}\|\delta w_j\|^2\ \le\alpha\|B_0\|^2\max_{j\in\{0,\dots,i\}}\|w_j\|^2\ \tau\sum_{j=1}^{i}\|\delta w_j\|^2$$

$$+\frac{5}{\epsilon}a\|B\|\max_{j\in\{0,\dots,i\}}\|w_j\|^2\tau\sum_{j=1}^{i}\|\delta F_j\|^2 \tag{58}$$

$$+\frac{5}{\epsilon}\tau\sum_{j=1}^{i}g_j^2\|w_0+\alpha B(B(w_0,w_0),w_j)\|^2+\frac{5}{\epsilon}\tau\sum_{j=1}^{i}\|\delta q_j\|^2$$

$$+\frac{5}{\epsilon}(1+2\|B\|\|B_0\|\max_{j\in\{0,\dots,i\}}\|w_j\|^2)\tau^3\sum_{j=2}^{i}\|\sum_{k=1}^{j-1}g_{j-k}\delta w_k\|^2.$$

The assumption (54) and uniform a priori estimates (49) imply the inequality

$$\alpha\|B_0\|^2\|w_j\|^2\le 1-\epsilon \text{ for }\ j=1,...,N. \tag{59}$$

Applying the assumption (59), the a priori estimate (49), the properties of the function g and the regularity assumptions $q \in W^{1,2}(0,T;V),\ F \in W^{1,2}(0,T;H^2(\Omega))$, we obtain the inequality

$$\tau \sum_{j=1}^{i} \|\delta w_j\|^2 \ \leq c_1 + c_2\tau^2 \sum_{j=1}^{i}\sum_{k=1}^{j-1} \|\delta w_k\|^2,\ i = 1, ...N. \tag{60}$$

The discrete Gronwall lemma [13] (Lemma 10.5) then implies the a priori estimate

$$\tau \sum_{j=1}^{N} \|\delta w_j\|^2 \ \leq C_3,\ \tau = \frac{T}{N}. \tag{61}$$

with a constant $C_3 \equiv C_3(\beta, \gamma, K, \epsilon, T)$.

In order to perform the convergence analysis we introduce the increasing sequence $\{N_n\},\ \lim_{n\to\infty} N_n = \infty$. We set

$$\begin{aligned}
&\tau_n = \frac{T}{N_n},\ t_i^n = i\tau_n,\ u_i^n = u(t_i^n),\ i = 0, 1, ..., N_n,\ u : [0,T] \to X,\\
&X\ -\ \text{any normed space },\\
&w_0^n = w_0,\ w_i^n = w_i,\ \delta w_j^n = \frac{1}{\tau}(w_i^n - w_{i-1}^n),\ i = 1, ..., N_n,
\end{aligned}$$

where $w_i \in V$ is a solution of the equation (41) with $\tau \equiv \tau_n,\ F_i \equiv F_i^n,\ g_{i-j} \equiv g_{i-j}^n,\ q_i \equiv q_i^n$.

Let us further define the following segmentline and step functions determined by values $w_i^n,\ \delta w_i^n$:

$$\begin{aligned}
&w_n : [0,T] \to V,\ w_n(t) = w_{i-1}^n + (t - t_i^n)\delta w_i^n,\ t_{i-1}^n \leq t \leq t_i^n,\\
&\bar{w}_n : [0,T] \to V,\ \bar{w}_n(0) = w_0,\ \bar{w}_n(t) = w_i^n,\ t_{i-1}^n < t \leq t_i^n,\\
&\tilde{w}_n : [0,T] \to V,\ \tilde{w}_n(0) = 0,\ \tilde{w}_n(t) = w_{i-1}^n,\ t_{i-1}^n < t \leq t_i^n,\\
&i = 1, ..., N_n.
\end{aligned}$$

The next theorem with its proof describes the convergence of the subsequence from $\{w_n\}$ to a solution w of the canonical integral equation (36). We shall verify also the unicity conditions.

Theorem 3.5 *Let the function $F : [0,T] \to H^2(\Omega)$ defined in (12), (13) fulfil the condition (38), the function $g \in C(R^+)$ fulfil the growth condition (37). Let $q \in W^{1,2}(0,T;V)$ fulfil the bound*

$$C_2(\beta, \gamma, K) \left(\int_0^t e^{-\gamma(t-s)} \|q(s)\|^2 ds\right)^{1/2} + \|q(t)\| < \frac{1}{\sqrt{\alpha}\|B_0\|} \forall t \in [0,T]. \tag{62}$$

Then there exists a solution $w \in W^{1,2}(0,T;V)$ *of the equation (36).*

If moreover

$$C_2(\beta,\gamma,K)\left(\int_0^t e^{-\gamma(t-s)}\|q(s)\|^2 ds\right)^{1/2} + \|q(t)\| < \frac{\sqrt{\beta}}{\sqrt{\alpha(\beta+K)}\|B_0\|} \quad (63)$$
$$\forall t \in [0,T],$$

then the solution $w \in W^{1,2}(0,T;V)$ *of (36) is unique.*

Proof. The assumption (62) implies that there exists such $\epsilon \in (0,1)$ and $\tau_0 > 0$ that condition (54) from Lemma 3.4 holds for every $\tau \in (0,\tau_0)$ and the a priori estimate

$$\tau_n \sum_{j=1}^{N_n} \|\delta w_j^n\|^2 \ \le C_4$$

holds.

The sequence of segmentline functions $\{w_n\}$ defined by their discrete values is then bounded in the space $W^{1,2}(0,T;V)$:

$$\|w_n\|_{W^{1,2}(0,T;V)} \le C_5,\ n \in N. \quad (64)$$

Then there exists its subsequence (again denoted by $\{w_n\}$) and a function $w \in W^{1,2}(0,T;V)$ such that

$$w_n \rightharpoonup w \qquad \text{in} \qquad W^{1,2}(0,T;V), \quad (65)$$
$$w_n(t) \rightharpoonup w(t),\ \bar{w}_n(t) \rightharpoonup w(t) \text{ in } V \ \text{ for every } t \in [0,T], \quad (66)$$
$$w_n \rightharpoonup^* w,\ \bar{w}_n \rightharpoonup^* w \text{ in } L^\infty(0,T;V), \quad (67)$$
$$w_n \to w,\ \bar{w}_n \to w \text{ in } L^p(0,T;W^{1,r}(\Omega)),\ p>1,\ r>1. \quad (68)$$

Let us introduce the discrete values of the Airy stress function Ψ by

$$\Psi_i^n = -\alpha D(0)[B_0(w_i^n,w_i^n) + \tau\sum_{j=0}^{i-1} g_{i-j}^n B_0(w_j^n,w_j^n)], \quad (69)$$
$$i = 1,...,N_n,\ n = 1,2,...$$

The corresponding sequence $\bar{\Psi}_n$ of step functions is due to the inquality (52) and the estimates (46), (49). bounded in the space $L^\infty(0,T;H_0^2(\Omega))$:

$$\|\bar{\Psi}_n\|_{L^\infty(0,T;H_0^2(\Omega))} \le C_6,\ n = 1,2,... \quad (70)$$

Then there exists a subsequence (again denoted by $\bar{\Psi}_n$) and a function $\Psi \in L^\infty(0,T;H_0^2(\Omega))$ such that

$$\bar{\Psi}_n \rightharpoonup^* \Psi \text{ in } L^\infty(0,T;H_0^2(\Omega)). \tag{71}$$

We shall verify that a function Ψ is determined by the expression

$$\Psi = -\alpha D(0)[B_0(w,w) + g * B_0(w,w)]. \tag{72}$$

Let us set

$$B_0(w,w) = U,\ B_0(w_n,w_n) = U_n,\ n = 1,2,...$$

We can express the functions $\bar{\Psi}_n$ in a following way:

$$\bar{\Psi}_n(t) = -\alpha D(0)\left[\bar{U}_n(t) + \int_0^t g(t-s)\tilde{U}_n(s)ds\right] + \tag{73}$$
$$\alpha D(0)\left[\int_t^{t_i^n} g(t-s)\tilde{U}_n(s)ds + \int_0^{t_i^n}(g(t-s) - g(t_i^n - s))\tilde{U}_n(s)ds\right],$$
$$t_{i-1}^n < t \le t_i^n,\ i = 1,...,N_n.$$

Applying the property (31), the convergence (68) and the boundedness of $\{\bar{w}_n\}$, $\{\tilde{w}_n\}$ in $L^\infty(0,T;V)$ and hence also in $L^\infty(0,T;W^{1,4}(\Omega))$ we obtain the convergence

$$\bar{U}_n \to U \text{ in } L^p(0,T;H_0^2(\Omega)), \tag{74}$$
$$\tilde{U}_n \to U \text{ in } L^p(0,T;H_0^2(\Omega))\ \forall p > 1. \tag{75}$$

The operator $G : L^p(0,T;H_0^2(\Omega)) \to L^p(0,T;H_0^2(\Omega))$ defined by

$$(Gu)(t) = \int_0^t g(t-s)u(s)ds,\ u \in L^p(0,T;H_0^2(\Omega))$$

is linear and continuous and the convergence

$$G\tilde{U}_n \to GU \text{ in } L^p(0,T;H_0^2(\Omega)) \tag{76}$$

follows.

The function defined by the sum of the second and the third integral in (73) converges strongly to 0 in $L^p(0,T;H_0^2(\Omega))$ as a consequence of previous a priori estimates and properties of the function g. Then we obtain using (74), (76) the relations (71), (72). Moreover we have the strong convergence

$$\bar{\Psi}_n \to \Psi \text{ in } L^p(0,T;H_0^2(\Omega)),\ p>1. \tag{77}$$

The equations (41) for $i=1,...,N_n$ can be expressed in a form

$$\bar{w}_n(t) - \frac{1}{D(0)}B(\bar{F}_n+\bar{\Psi}_n,\bar{w}_n)(t) + G\tilde{w}_n(t) + \tag{78}$$
$$\int_t^{t_i^n} g(t-s)\tilde{w}_n(s)ds + \int_0^{t_i^n}(g(t-s)-g(t_i^n-s))\tilde{w}_n(s)ds = q_n(t),$$
$$t_{i-1}^n < t \le t_i^n,\ i=1,...,N_n.$$

Applying the convergence (68), (77), the regularity of the functions $F: [0,T]\to H^2(\Omega)$, $q:[0,T]\to V$ and the relation (72) we obtain in the same way as above that the function w fulfils the canonical equation (36).

Let the assumption (63) hold. We shall verify the uniqueness of a solution which implies that the convergence (65)-(68), (71), (77) holds for the whole sequence $\{w_n,\bar{\Psi}_n\}$.

Let w_1 and w_2 be two solutions of the equation (36). They fulfil the equations

$$w_i(t) - aB(F(t),w_i(t)) + (g*w_i)(t) + \tag{79}$$
$$\alpha B[B_0(w_i,w_i)(t) + (g*B_0(w_i,w_i))(t), w_i(t)] = q(t),\ t\in[0,T].$$

The difference $u=w_2-w_1$ then fulfils the equation

$$\begin{aligned}&u(t) - aB(F(t),u(t)) + (g*u)(t)\\ &+\alpha B\left(B_0(w_2,w_2)(t) + (g*B_0(w_2,w_2))(t), w_2(t)\right)\\ &-\alpha B\left(B_0(w_1,w_1)(t) + (g*B_0(w_1,w_1))(t), w_1(t)\right) = 0.\end{aligned}$$

Let $w_\xi = w_1+\xi(w_2-w_1),\ \xi\in R$. There hold the relations

$$\begin{aligned}&((\ B(B_0(w_2,w_2),w_2)(t) - B(B_0(w_1,w_1),w_1)(t), u(t)\ ))\\ &= \int_0^1 [2\|B_0(u(t),w_\xi(t))\|_0^2 + ((\ B_0(w_\xi,w_\xi)(t), B_0(u(t),u(t))\ ))_0]d\xi,\end{aligned} \tag{80}$$

$$\begin{aligned}&((\ B(g*B_0(w_2,w_2))(t),w_2(t)) - B(g*B_0(w_1,w_1))(t),w_1)(t)), u(t)\ ))\\ &= 2\int_0^1 ((\ g*B_0(u,w_\xi)(t)), B_0(u,w_\xi)(t)\ ))_0 d\xi\\ &+\int_0^1 ((\ (g*B_0(w_\xi,w_\xi))(t), B_0(u,u)(t)\ ))_0 d\xi.\end{aligned} \tag{81}$$

Using the assumption (38) and the relations (80), (81) we obtain the inequality

$$\|u\|^2 + ((\int_0^t g(t-s)u(s)ds, u(t)))$$
$$+\alpha \int_0^1 [2\|B_0(u(t), w_\xi(t))\|_0^2 + ((\ B_0(w_\xi, w_\xi)(t), B_0(u,u)(t)\ ))_0]d\xi$$
$$+\alpha \int_0^1 [2((\int_0^t g(t-s)B_0(u, w_\xi)(s)ds, B_0(u, w_\xi)(t)\ ))_0$$
$$+((\int_0^t g(t-s)B_0(w_\xi, w_\xi)(s)ds, B_0(u,u)(t)\ ))_0\ ]d\xi \le 0.$$

After applying the growth assumption (37) we arrive at the inequality with an arbitrary $\epsilon > 0$:

$$(1-\epsilon)[\|u(t)\|^2 + 2\alpha \int_0^1 \|B_0(u(t), w_\xi(t))\|^2 d\xi] \le$$
$$\alpha\|B_0\|^2(1+\frac{K}{\beta}) \max\{\max_{t\in[0,T]} \|w_1(t)\|^2, \max_{t\in[0,T]} \|w_2(t)\|^2\} +$$
$$C(\epsilon) \int_0^t [\|u(s)\|^2 + 2\alpha \int_0^1 \|B_0(u(s), w_\xi(s))\|^2 d\xi] ds \quad \text{for all } t \in [0,T]$$

Using the same approach as in the discrete case (Lemma 3.2, Lemma 3.3) the estimates

$$\|w_i(t)\| \le C_2(\beta,\gamma,K)(\int_0^t e^{-\gamma(t-s)}\|q(s)\|^2 ds)^{1/2} + \|q(t)\|, \qquad (82)$$
$$t \in [0,T],\ i = 1,2 \qquad (83)$$

can be derived. The assumption (63) then implies

$$\|w_i(t)\|^2 < [\alpha\|B_0\|^2(1+\frac{K}{\beta})]^{-1} \text{ for all } t \in [0,T],\ i = 1,2. \qquad (84)$$

Then there exists such $\epsilon > 0$ that there holds the inequality

$$[\|u(t)\|^2 + 2\alpha \int_0^1 \|B_0(u(t), w_\xi(t))\|^2 d\xi] \le$$
$$\frac{C(\epsilon)}{\epsilon} \int_0^t [\|u(s)\|^2 + 2\alpha \int_0^1 \|B_0(u(s), w_\xi(s))\|^2 d\xi] ds \quad \text{for all } t \in [0,T]$$

and the uniqueness of a solution follows after applying the Gronwall lemma.

Remark 3.6 *If $F \in W^{1,\infty}(0,T;H^2(\Omega))$ and $q \in W^{1,\infty}(0,T;V)$ then we obtain in the same way as in Lemma 3.4 the a priori estimate of the norms $\|\delta w_i^n\|$ and the boundedness of the sequence $\{w_n\}$ in $W^{1,\infty}(0,T;V)$. In this case is the condition (62) sufficient both for the existence and the uniqueness of a solution $w \in W^{1,\infty}(0,T;V)$.*

Applying Theorem 3.5 we obtain directly a theorem on the existence and the uniqueness of a weak solution of the original system (1)-(6).

Theorem 3.7 *Let the function $F : [0,T] \to H^2(\Omega)$ defined in (12), (13) fulfil the condition (39) and the positive relaxation function $E \in C^1(R^+)$ fulfil the growth condition*

$$0 < -E'(t) \leq KE(0)e^{-\beta t},\ t \geq 0,\ 0 < K < \beta - \gamma,\ \gamma > 0.$$

Let

$$f \in W^{1,2}(0,T;V^*),$$
$$m_i \in W^{1,2}(0,T;L^p(\Gamma_i)),\ i = 2,3,\ t_3 \in W^{1,2}(0,T;L^p(\Gamma_3)).$$

Let a linear continuous functional $\mathcal{L}(t) \in V^*$ defined by

$$\langle \mathcal{L}(t), v\rangle =$$
$$\int_{\Gamma_2} m_2(t)\frac{\partial v}{\partial \nu}d\sigma + \int_{\Gamma_3}(m_3(t)\frac{\partial v}{\partial \nu} + t_3(t)v)d\sigma + \langle f(t), v\rangle\ \forall v \in V$$

fulfil the condition

$$C_2(\beta,\gamma,K)\left(\int_0^t e^{-\gamma(t-s)}\|\mathcal{L}(s)\|_*^2 ds\right)^{1/2} + \|\mathcal{L}(t)\|_* < \frac{D(0)}{\sqrt{\alpha}\|B_0\|}\forall t \in [0,T].$$

Then there exists a weak solution

$$\{w,\Phi\} = \{w, \Psi + F\} \in W^{1,2}(0,T;V) \times C([0,T];H^2(\Omega)),$$
$$\Psi(t) = -\frac{h}{2}[E(0)B_0(w,w)(t) + E' * B_0(w,w)(t)],\ t \in [0,T].$$

of the von Kármán system (1)- (6).

If moreover

$$C_2(\beta,\gamma,K)\left(\int_0^t e^{-\gamma(t-s)}\|\mathcal{L}(s)\|_*^2 ds\right)^{1/2} + \|\mathcal{L}(t)\|_* < \frac{D(0)\sqrt{\beta}}{\sqrt{\alpha(\beta+K)}\|B_0\|}$$
$$\forall t \in [0,T],$$

then a solution $\{w,\Phi\} \in W^{1,2}(0,T;V) \times C([0,T];H^2(\Omega))$ is unique.

4 The Behaviour of a Solution for $t \to \infty$

We have verified in [1] that the conditions

$$\lim_{t\to\infty} q(t) = q_\infty \lim_{t\to\infty} q'(t) = 0 \text{ in } H_0^2(\Omega)$$

imply that a solution w of the nonlinear pseudoparabolic problem

$$\begin{aligned} & w'(t) + aw(t) + bB(B(w,w)' + aB(w,w),w)(t) = \\ & q'(t) + \beta q(t), \qquad (85) \\ & w(0) + bB(B(w(0),w(0)),w(0)) = q(0) \qquad (86) \end{aligned}$$

fulfils the limit behaviour

$$\lim_{t\to\infty} w(t) = w_\infty \text{ in } H_0^2(\Omega), \qquad (87)$$

where w_∞ is a solution of the stationary problem

$$aw_\infty + abB(B(w_\infty,w_\infty),w_\infty) = q_\infty.$$

The initial value problem (85), (86) is equivalent with the problem (36) if we set

$$V = H_0^2(\Omega),\ F(t) \equiv 0,\ B_0 = B,\ E(t) = E_0 + \beta E_1 e^{-\beta t}.$$

The stationary problem (86) can be expressed in a form

$$D_\infty w_\infty + \frac{hE_\infty}{2} B(B(w_\infty,w_\infty),w_\infty) = D(0)q_\infty.$$

This behaviour of a solution w for large values of the time variable can be verified for the problem (36) with a general relaxation function E. We assume only the bound on the limit right-hand side q_∞.

Theorem 4.1 *Let $q \in C([0,\infty),V)$ and $q_\infty \in V$ be such that*

$$\lim_{t\to\infty} q(t) = q_\infty, \qquad (88)$$

$$\|q_\infty\| \le \frac{\sqrt{\gamma(2\beta-\gamma)}}{\sqrt{2\alpha}\beta\|B_0\|} C_7(\beta,\gamma,K), \qquad (89)$$

$$C_7(\beta,\gamma,K) = \left[1 + \frac{\beta(\beta+K)}{2(\beta-K)^2} + \frac{K}{\gamma}\left(\frac{8K}{2\beta-\gamma} + \frac{1}{2}\right) C_1(\beta,\gamma,K)\right]^{-1/2}.$$

Let the functions $E \in C^1([0,\infty);R),\ F \in C([0,\infty),H^2(\Omega))$ *fulfil*

$$0 < -E'(t) \le KE(0)e^{-\beta t},\ t \ge 0,\ 0 < K < \beta - \gamma,\ \gamma > 0, \tag{90}$$

$$\lim_{t\to\infty} E(t) = E_\infty, \tag{91}$$

$$((B(F(t),v),v)) \le 0 \text{ for all } t \in [0,\infty) \text{ and } v \in V, \tag{92}$$

$$\lim_{t\to\infty} F(t) = F_\infty \text{ in } H^2(\Omega). \tag{93}$$

If $w \in C([0,\infty),V)$ *is a solution of the equation*

$$\begin{aligned} &D(0)w(t) + D' * w(t) - B(F(t),w(t)) + \\ &\frac{h}{2}[B(E(0)B_0(w,w)(t) + E' * B_0(w,w)(t), w(t))] = D(0)q(t), \end{aligned} \tag{94}$$

then

$$\lim_{t\to\infty} w(t) = w_\infty \text{ in } V, \tag{95}$$

where $w_\infty \in V$ *satisfies the equation*

$$D_\infty w_\infty - B(F_\infty, w_\infty) + \frac{h}{2}E_\infty B(B_0(w_\infty,w_\infty),w_\infty) = D(0)q_\infty. \tag{96}$$

Proof. The existence and uniqueness of the function $w_\infty \in V$ is assured due to the theory of stationary von Kármán equations [5]. Let us set

$$u(t) = w(t) - w_\infty. \tag{97}$$

The function $u \in C([0,\infty),V)$ fulfils the identity

$$\begin{aligned} &D(0)u(t) + D' * u(t) - B(F(t),u(t)) \\ &+\frac{h}{2}E(0)[B(B_0(w,w)(t),w(t)) - B(B_0(w_\infty,w_\infty),w_\infty)] \\ &+\frac{h}{2}[B(E' * B_0(w,w)(t),w(t)) - B(E' * B_0(w_\infty,w_\infty),w_\infty)] = r(t), \end{aligned} \tag{98}$$

where

$$r(t) = D(0)[q(t) - q_\infty] - [D(t) - D_\infty]w_\infty + B(F(t) - F_\infty, w_\infty). \tag{99}$$

Let us set $u_\xi(t) = w_\infty + \xi u(t)$. Applying the analogous relations as (80), (81) we obtain after multiplying the relation (98) with $\frac{1}{D(0)}u(t)$ in the space

V the inequality

$$\begin{aligned}
&\|u(t)\|^2 + ((\int_0^t g(t-s)u(s)ds, u(t))) + \\
&\alpha \int_0^1 [2\|B_0(u(t), u_\xi(t))\|_0^2 + ((B_0(u_\xi, u_\xi)(t), B_0(u(t), u(t))))_0] d\xi + \\
&\alpha \int_0^1 [2((\int_0^t g(t-s)B_0(u, u_\xi)(s)ds, B_0(u, u_\xi)(t)))_0 + \\
&((\int_0^t g(t-s)B_0(u_\xi, u_\xi)(s)ds, B_0(u, u)(t)))_0 \,] d\xi \le \\
&((r(t), u(t))), \ t \ge 0.
\end{aligned}$$

Let us denote

$$\omega(t) = \|u(t)\|^2 + 2\alpha \int_0^1 \|B_0(u(t), u_\xi(t))\|_0^2 d\xi, \ t \ge 0.$$

We obtain for arbitrary $\epsilon \in (0,1)$ the inequality

$$\begin{aligned}
&\frac{1-\epsilon}{2}\omega(t) \le \frac{1}{2}\left[\|g * u(t)\|^2 + 2\alpha \int_0^1 \|g * B_0(u(t), u_\xi)(t)\|_0^2 d\xi\right] \\
&+\alpha\|B_0\|^2 \int_0^1 [u_\xi(t)\|^2 + K \int_0^t e^{-\beta(t-s)}\|u_\xi(s)\|^2 ds] d\xi \ \|u(t)\|^2 \\
&+\frac{1}{2\epsilon}\|r(t)\|^2, \ t \ge 0.
\end{aligned}$$

Let us assume that there holds for any $T \ge 0$ and $\delta \in (0, 1-\epsilon)$ the estimate

$$\alpha\|B_0\|^2 \int_0^1 [\|u_\xi(t)\|^2 + K \int_0^t e^{-\beta(t-s)}\|u_\xi(s)\|^2 ds] d\xi < \frac{\delta}{2}, \ t \ge T. \tag{100}$$

Applying the exponential growth assumption (90) and the Cauchy-Schwarz inequality in the convolution integrals we obtain the inequality

$$\omega(t) \le \frac{K^2}{\beta(1-\delta-\epsilon)} \int_0^t e^{-\beta(t-s)}\omega(s)ds + \frac{1}{\epsilon(1-\delta-\epsilon)}\|r(t)\|^2, \ t \ge T. \tag{101}$$

Let us set

$$e^{\beta t}\omega(t) - \frac{K^2}{\beta(1-\delta-\epsilon)} \int_0^t e^{\beta s}\omega(s)ds = \rho(t), \ t \ge 0 \tag{102}$$

with a function $\rho \in C([0,\infty))$ fulfilling the inequality

$$\rho(t) \leq \frac{1}{\epsilon(1-\delta-\epsilon)} \|r(t)\|^2 \ \forall \ t > T. \tag{103}$$

After solving the ordinary differential equation (102) we obtain

$$\int_0^t e^{\beta s} \omega(s) ds = \int_0^t e^{\frac{K^2}{\beta(1-\delta-\epsilon)}(t-s)} \rho(s) ds$$

and

$$\omega(t) = e^{-\beta t} \left[\rho(t) + \frac{K^2}{\beta(1-\delta-\epsilon)} \int_0^t e^{\frac{K^2}{\beta(1-\delta-\epsilon)}(t-s)} \rho(s) ds \right].$$

The inequality (103) implies the estimate

$$\begin{aligned} \omega(t) \ &\leq \ \frac{1\|r(t)\|^2}{\epsilon(1-\delta-\epsilon)} + \frac{K^2}{\beta(1-\delta-\epsilon)} e^{-\kappa t} \int_0^T e^{-\frac{K^2}{\beta(1-\delta-\epsilon)} s} \rho(s) ds \\ &+ \ \frac{K^2}{\beta\epsilon(1-\delta-\epsilon)^2} \int_T^t e^{-\kappa(t-s)} \|r(s)\|^2 ds \ \forall \ t > T, \qquad (104) \\ &\kappa = \beta - \frac{K^2}{\beta(1-\delta-\epsilon)}. \end{aligned}$$

The assumptions (88), (91), (93) imply

$$\lim_{t\to\infty} r(t) = 0 \text{ in } V. \tag{105}$$

In order to obtain the limit

$$\lim_{t\to\infty} \omega(t) = 0 \tag{106}$$

we need to find such value of δ that

$$\beta - \frac{K^2}{\beta(1-\delta)} > 0 \tag{107}$$

In this case there exists such $\epsilon \in (0,1)$ that

$$\beta - \frac{K^2}{\beta(1-\delta-\epsilon)} > 0$$

and hence $\kappa > 0$. Setting

$$\delta = \frac{\gamma(2\beta-\gamma)}{\beta^2} \tag{108}$$

we obtain $(1-\delta)\beta^2 = (\beta-\gamma)^2$ and the inequality (107) follows due to the assumption (90). The convergence (106) implies $\lim_{t\to\infty} \|u(t)\|^2 = 0$ and the assertion (95) of the theorem holds.

Hence it remains us to verify for some $T > 0$ the bound (100) with a constant δ determined by (108). Let us first estimate the left-hand side of (100). We have the inequality

$$\int_0^1 [u_\xi(t)\|^2 + K\int_0^t e^{-\beta(t-s)}\|u_\xi(s)\|^2 ds]d\xi$$
$$\leq \frac{1}{2}\|w(t)\|^2 + \frac{1}{2}(1+\frac{K}{\beta})\|w_\infty\|^2 + \frac{K}{2}\int_0^t e^{-\gamma(t-s)}\|w(s)\|^2 ds,\ t \geq 0.$$

Applying the continuous analogy of the estimates (46), (49) and the estimate

$$\frac{D(0)}{D_\infty} \leq \frac{\beta}{\beta - K}$$

we obtain

$$\frac{1}{2}\|w(t)\|^2 + \frac{1}{2}(1+\frac{K}{\beta}) + \frac{K}{2}\int_0^t e^{-\gamma(t-s)}\|w(s)\|^2 ds \leq$$
$$\|q(t)\|^2 + \frac{\beta(\beta+K)}{2(\beta-K)^2}\|q_\infty\|^2 +$$
$$K\left(\frac{8K}{2\beta-\gamma} + \frac{1}{2}\right) C_1(\beta,\gamma,K)\int_0^t e^{-\gamma(t-s)}\|q(t)\|^2 ds,\ t \geq 0.$$

The limit (88) implies

$$\lim_{t\to\infty}\int_0^t e^{-\gamma(t-s)}\|q(t)\|^2 ds = \frac{1}{\gamma}\|q_\infty\|^2.$$

The estimate (89) then implies the existence of such $T > 0$ that the estimate (100) holds with the constant δ defined by (108) and the proof is completed.

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