BLOCK-FINITE EFFECT ALGEBRAS AND THE EXISTENCE OF STATES

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Abstract. Lattice effect algebras generalize orthomodular lattices and MV-algebras in the quantum or fuzzy probability theory. Every lattice effect algebra E is a union of its maximal MV-effect subalgebras called blocks of E. We show that an Archimedean lattice effect algebra with exactly two blocks is either a horizontal sum of two blocks or (up to isomorphism) a direct product of an MV-effect algebra and a horizontal sum of two blocks. Further, every complete effect algebra with nontrivial center and finitely many blocks is isomorphic to a direct product of an MV-effect algebra M (it may be $M = \{0\}$) and finitely many effect algebras with trivial centers and at least two blocks each. As corollaries we obtain the existence of states or order-continuous subadditive states (probabilities) on some complete or Archimedean effect algebras with nontrivial center and finitely many blocks.

1. Introduction and basic definitions

In recent years effect algebras [2] or equivalent in some sense D-posets [10], [11] have been studied as carriers of states or probability measures in the quantum or fuzzy probability theory.

Effect algebras have been introduced by Foulis and Bennet [2] as an algebraic structure providing an instrument for studying quantum effects that may be unsharp. Kôpka [10] introduced a D-poset of fuzzy sets in which the operation of difference of fuzzy sets is the primary operation. For the connection between effect algebras and D-posets we refer to [1] and [12].

Definition 1.1. A structure $(E; \oplus, 0, 1)$ is called an *effect-algebra* if 0, 1 are two distinguished elements and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a, b, c \in E$:
(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,
(Eiii) for every $a \in E$ there exists a unique $b \in E$ such that $a \oplus b = 1$ (we put $a' = b$),
(Eiv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. In every effect algebra E we can define the partial operation $\ominus$ and the partial order $\leq$ by putting

$$a \leq b \text{ and } b \ominus a = c \text{ iff } a \oplus c \text{ is defined and } a \oplus c = b\,.$$

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Since $a \oplus c = a \oplus d$ implies $c = d$, the $\ominus$ and the $\leq$ are well defined. If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete effect algebra*). For more details we refer the reader to [1] and the references given there.

Definition 1.2. Let $(E; \oplus, 0, 1)$ be an effect algebra. $Q \subseteq E$ is called a *sub-effect algebra of E* iff

(i) $1 \in Q$,

(ii) if $a, b, c \in E$ with $a \oplus b = c$ and out of a, b, c at least two elements are in Q then $a, b, c \in Q$.

Note that if Q is a sub-effect algebra of E then Q with inherited operation $\oplus$ is an effect algebra in its own right.

Recall that elements a, b of a lattice effect algebra $(E; \oplus, 0, 1)$ are called *compatible* (written $a \leftrightarrow b$) iff $a \vee b = a \oplus (b \ominus (a \wedge b))$ (see [11]). $P \subseteq E$ is a *set of pairwise compatible elements* if $a \leftrightarrow b$ for all $a, b \in P$. $M \subseteq E$ is called a *block* of E iff M is a maximal subset of pairwise compatible elements. Every block of a lattice effect algebra E is a sub-effect algebra and a sub-lattice of E and E is a union of its blocks (see [13]). Lattice effect algebra with a unique block is called an *MV-effect algebra*. Every block of a lattice effect algebra is an MV-effect algebra in its own right. In [13] it was proved that *every block M of a lattice effect algebra E is closed with respect to all existing infima and suprema of subsets of M*. We say that M is a *full sub-lattice* of E.

A lattice effect algebra E is a *horizontal sum* of blocks if $A \cap B = \{0, 1\}$ holds for every pair of its blocks A and B.

A nonzero element a of an effect algebra E is called an *atom* if $0 \leq b < a$ implies $b = 0$. E is called atomic if for every nonzero element $x \in E$ there is an atom $a \in E$ with $a \leq x$.

An effect algebra E is called *Archimedean* if for no nonzero element $e \in E$, $ne = e \oplus e \oplus \cdots \oplus e$ (n times) exists for all positive integer n. We write $\mathrm{ord}(e) = n_e \in N$ if n_e is the greatest integer such that $n_e e$ exists in E. Every complete effect algebra is Archimedean, [20].

Definition 1.3. Let $(E; \oplus_E, 0_E, 1_E)$ and $(F; \oplus_F, 0_F, 1_F)$ be effect algebras. A bijective map $\varphi\colon E \to F$ is called an *isomorphism* if

(i) $\varphi(1_E) = 1_F$,

(ii) for all $a, b \in E$: $a \leq b'$ iff $\varphi(a) \leq \big(\varphi(b)\big)'$ in which case $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$.

We write $E \cong F$. Sometimes we identify E with $F = \varphi(E)$.

Definition 1.4. A map $\omega\colon E \to [0, 1] \subseteq R$ is called a *state* if (i) $\omega(1) = 1$, and (ii) if $a, b \in E$ with $a \leq b'$ then $\omega(a \oplus b) = \omega(a) + \omega(b)$. A state ω is called *order-continuous* if for every net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E, $x_\alpha \downarrow 0 \implies \omega(x_\alpha) \downarrow 0$. Here $x_\alpha \downarrow 0$ means that $x_{\alpha_1} \leq x_{\alpha_2}$ for all $\alpha_2 \geq \alpha_1$, $\alpha_1, \alpha_2 \in \mathcal{E}$ and $\bigwedge\{x_\alpha \mid \alpha \in \mathcal{E}\} = 0$.

Lattice effect algebras generalize orthomodular lattices and MV-algebras (including Boolean algebras). Unfortunately, there are even finite (lattice) effect algebras admitting no states, [4], [18]. Some positive results on the existence of states or subadditive (o)-continuous states (probabilities) were given in [16], [17] and [19].

In this paper we prove the existence of states or probabilities on some effect algebras with finitely many blocks, called *block-finite*. Motivations for that were the results on decompositions of block-finite orthomodular lattices due to G. Bruns and representations of finite orthomodular lattices by Greechie diagrams. In a Greechie diagram of an orthomodular lattice L, the points are in one-to-one correspondence with atoms of L and the lines are in one-to-one correspondence with blocks of L. We refer the reader to the book [9], pp. 40–64 and 297–307 and references given there.

The following lemma will be often used in the sequel.

Lemma 1.5 [8]**.** *In a lattice effect algebra E, for $A \subseteq E$ and $b \in E$ such that $\bigvee A$ exists and $b \leftrightarrow a$ for all $a \in A$, the following conditions are satisfied:*
(i) $b \leftrightarrow \bigvee A$,
(ii) $\bigvee\{b \wedge a \mid a \in A\}$ *exists,*
(iii) $b \wedge \bigvee A = \bigvee\{a \wedge b \mid a \in A\}$.

2. Effect algebras with two blocks and trivial center

An element z of an effect algebra E is called *central* if $x = (x \wedge z) \vee (x \wedge z')$ for all $x \in E$. The *center* $C(E)$ of E is the set of all central elements of E, [5]. If E is lattice ordered then $z \in E$ is central iff $z \wedge z' = 0$ and $z \leftrightarrow x$ for all $x \in E$, [12]. Thus in a lattice effect algebra $C(E) = B(E) \cap S(E)$, where $B(E) = \bigcap\{M \subseteq E \mid E \text{ is a block of } E\}$ is the *compatibility center* of E and $S(E) = \{x \in E \mid x \wedge x' = 0\}$ is the set of *sharp elements* of E. Evidently, $B(E) = \{x \in E \mid x \leftrightarrow y \text{ for all } y \in E\}$. Moreover, if E is complete then every block of E is complete and hence $B(E)$ is a complete MV-effect algebra. Further, in every complete effect algebra E, $S(E)$ is a complete orthomodular lattice and hence $C(E)$ is a complete Boolean algebra, [15].

Theorem 2.1. *If a lattice effect algebra E can be covered by two blocks then E contains exactly two blocks.*

Proof. Let $E = M_1 \cup M_2$, where $M_1 \neq M_2$ are blocks of E. Assume to the contrary that there exists a block M of E such that $M \neq M_1$ and $M \neq M_2$. By maximality of blocks, $M \not\subseteq M_1$ and $M \not\subseteq M_2$ and hence there are elements $x \in M \setminus M_2$, $y \in M \setminus M_1$ which gives $x \in M_1 \setminus M_2$, $y \in M_2 \setminus M_1$ and $x \leftrightarrow y$. It follows that $x, y \notin \{0, 1\}$ and $x \vee y = x \oplus (y \ominus (x \wedge y)) = y \oplus (x \ominus (x \wedge y))$.

Assume that $x \wedge y \notin M_2$. Then $y \ominus (x \wedge y) \notin M_2$, because otherwise $x \wedge y = y \ominus (y \ominus (x \wedge y)) \in M_2$, which contradicts to the assumption. It follows that $y = (x \wedge y) \oplus (y \ominus (x \wedge y)) \in M_1$, a contradiction. Thus $x \wedge y \in M_2$.

Assume that $x \vee y \notin M_2$. Then $(x \vee y) \ominus y \notin M_2$ because otherwise $x \vee y = ((x \vee y) \ominus y) \oplus y \in M_2$, which contradicts to the assumption. It follows that $x \ominus (x \wedge y) = (x \vee y) \ominus y \in M_1$ and hence $y = (x \vee y) \ominus (x \ominus (x \wedge y)) \in M_1$, a contradiction. Thus $x \vee y \in M_2$.

We obtain that $x \wedge y, x \vee y \in M_2$ which gives $x = (x \vee y) \ominus (y \ominus (x \wedge y)) \in M_2$, a contradiction. We conclude that $M = M_1$ or $M = M_2$.

Lemma 2.2. *Let E be a complete effect algebra. If $E = M_1 \cup M_2$, where $M_1 \neq M_2$ are blocks of E and $C(E) = \{0, 1\}$ then $M_1 \cap M_2 = \{0, 1\}$.*

Proof. Suppose, contrary to our claim, that $M_1 \cap M_2 \neq \{0,1\}$. Then for every $w \in (M_1 \cap M_2) \setminus \{0,1\}$ we have $w \wedge w' \neq 0$ because otherwise $w \in M_1 \cap M_2 \cap S(E) = C(E) = \{0,1\}$. Let $u \in M_1 \setminus M_2$ and $v \in M_2 \setminus M_1$. Clearly, every chain is in a block of E. Since

$$u \wedge (w \wedge w') \leq w \wedge w' \leq v \vee (w \wedge w') \quad \text{and} \quad u \wedge (w \wedge w') \leq w \wedge w' \leq w \vee w' \leq v' \vee (w \wedge w')'$$

the assumption $u \wedge (w \wedge w') \notin M_2$ implies that $v \vee (w \wedge w') \in M_1$ and $v' \vee (w \wedge w')' \in M_1$, which gives $v \wedge (w \wedge w') \in M_1$. It follows that $v = (v \vee (w \wedge w')) \ominus ((w \wedge w') \ominus (w \wedge w') \wedge v) \in M_1$, a contradiction. Hence $u \wedge (w \wedge w') \in M_2$. In the same manner we can show that $v \wedge (w \wedge w') \in M_1$. We conclude that $[0, w \wedge w'] \subseteq M_1 \cap M_2$. Put $A = \{x \in M_1 \cap M_2 \mid [0,x] \subseteq M_1 \cap M_2\}$ and $d = \bigvee A$. Then $d \in M_1 \cap M_2$, $d \neq 0$ and, by Lemma 1.5, for every $y \leq d$ we have $y = y \wedge d = \bigvee \{x \wedge y \mid [0,x] \subseteq M_1 \cap M_2\} \in M_1 \cap M_2$. If $d = 1$ then $E = M_1 \cap M_2$, a contradiction. Assume that $d < 1$. Then $d \wedge d' \neq 0$ and $[0, d \wedge d'] \subseteq M_1 \cap M_2$, as we have shown above. Further, for every $y \in E$ with $y \leq d \oplus (d \wedge d')$ we have either $\{y, d, d \wedge d'\} \subseteq M_1$ or $\{y, d, d \wedge d'\} \subseteq M_2$. By Riesz decomposition property (see [1]) there are $u \leq d$, $v \leq d \wedge d'$ such that $y = u \oplus v$. It follows that $y \in M_1 \cap M_2$, which gives $d \oplus (d \wedge d') \leq d$ and hence $d \wedge d' = 0$. We obtain that $d \in C(E)$ and $d \notin \{0,1\}$, a contradiction. We conclude that $M_1 \cap M_2 = \{0,1\}$.

For a central element d of a lattice effect algebra E the interval $[0,d]$ is a lattice effect algebra with the unit d and the partial operation $\oplus$ restricted from E. It is because for $x, y \leq d$ with $x \oplus y$ defined in E we have $x \oplus y \leq d$. Moreover, $d = (x \oplus x') \wedge d = (x \wedge d) \oplus (x' \wedge d)$ for all $x \in E$ which for $y \leq d$ implies $d = y \oplus (y' \wedge d)$, [17].

Lemma 2.3. *Let E be a lattice effect algebra.*

(i) *If $d \in E$ is an atom of $C(E)$ then $[0,d]$ is irreducible.*
(ii) *If for $d_1, d_2 \in C(E)$ the intervals $[0,d_1]$ and $[0,d_2]$ are MV-effect algebras then $[0, d_1 \vee d_2]$ is an MV-effect algebra.*
(iii) *If $A = \{d \in C(E) \mid d \neq 0$ and $[0,d]$ is an MV-effect algebra$\} \neq \emptyset$ and $\bigvee A$ exists in E then $w = \bigvee A \in C(E)$ and $[0,w]$ is an MV-effect algebra.*

Proof. (i) As for every $x \leq d$ we have $d = (x \oplus x') \wedge d = (x \wedge d) \oplus (x' \wedge d) = x \oplus (x' \wedge d)$, we obtain that $x \oplus x^* = d$ iff $x^* = x' \wedge d$. Thus $x \wedge x^* = x \wedge x' \wedge d = x \wedge x'$, which gives that $x \wedge x^* = 0$ iff $x \wedge x' = 0$. Hence $S([0,d]) = S(E) \cap [0,d]$. Further, for $x, y \leq d$ we have $x \leftrightarrow y$ in $[0,d]$ iff $x \leftrightarrow y$ in E, because $[0,d]$ is a sub-lattice of E with $\oplus$ and $\ominus$ inherited from E. Hence $B([0,d]) = B(E) \cap [0,d]$. We obtain that $C([0,d]) = C(E) \cap [0,d] = \{0,d\}$.

(ii) As $C(E) = B(E) \cap S(E) \subseteq B(E)$ we obtain that $d_1 \leftrightarrow d_2$ which gives $d_1 \vee d_2 = d_1 \oplus (d_2 \ominus (d_1 \wedge d_2)) = d_1 \vee (d_2 \ominus (d_1 \wedge d_2))$ because $C(E)$ is a Boolean algebra. By Lemma 1.5, for all $x, y \leq d_1 \vee d_2$ we have $x = x \wedge (d_1 \vee d_2) = (x \wedge d_1) \vee (x \wedge (d_2 \ominus (d_1 \wedge d_2)) = (x \wedge d_1) \oplus (x \wedge (d_2 \ominus (d_1 \wedge d_2))$ and similarly $y = (y \wedge d_1) \oplus (y \wedge (d_2 \ominus (d_1 \wedge d_2))$. Moreover, $(x \wedge d_1) \oplus (y \wedge (d_2 \ominus (d_1 \wedge d_2))$ and $(y \wedge d_1) \oplus (x \wedge (d_2 \ominus (d_1 \wedge d_2))$ exist, because $d_1 \oplus (d_2 \ominus (d_1 \wedge d_2))$ exists. We conclude that the set $\{x \wedge d_1, y \wedge d_1, x \wedge (d_2 \ominus (d_1 \wedge d_2)), y \wedge (d_2 \ominus (d_1 \wedge d_2))\}$ is pairwise compatible, which implies that $x \leftrightarrow y$.

(iii) Since $C(E)$ is a full sub-lattice of E we have $w = \bigvee A \in C(E)$. Further, for all $x, y \in [0,w]$ we have $x = x \wedge w = \bigvee \{x \wedge d \mid d \in A\}$ and $y = y \wedge w = \bigvee \{y \wedge d \mid d \in A\}$. Since, by (ii), $x \wedge d_1 \leftrightarrow y \wedge d_2$ for all $d_1, d_2 \in A$, we obtain, by Lemma 1.5, that $x \leftrightarrow y$. This proves that $[0,w]$ is an MV-effect algebra.

Theorem 2.4. *Every Archimedean lattice effect algebra with two blocks and trivial center is a horizontal sum of its blocks.*

Proof. Assume that E is an Archimedean lattice effect algebra, $E = M_1 \cup M_2$, where M_1 and M_2 are two different blocks of E and $C(E) = \{0, 1\}$. By [20], up to isomorphism, E is a supremum-dense sub-effect algebra of a complete effect algebra $\widehat{E}$, being a MacNeille completion of E. Further, there are blocks $\widehat{M_1}$ and $\widehat{M_2}$ of $\widehat{E}$ such that $M_1 \subseteq \widehat{M_1}$, $M_2 \subseteq \widehat{M_2}$ and $\widehat{E} = \widehat{M_1} \cup \widehat{M_2}$.

Assume that $C(\widehat{E}) \neq \{0, 1\}$. Then there is $w \in C(\widehat{E}) \setminus \{0, 1\}$. By [5], $\widehat{E} \cong [0, w] \times [0, w']$. Because, by Theorem 2.1, $\widehat{E}$ has exactly two blocks, we obtain that one of the effect algebras $[0, w]$ and $[0, w']$ has a unique block and the other has two blocks. Put $z = \bigvee\{w \in C(\widehat{E}) \mid [0, w] \text{ is an MV-effect algebra}\}$. Then $z \in C(\widehat{E}) \setminus \{0, 1\}$ and $\widehat{E} \cong [0, z] \times [0, z']$. Moreover, by Lemma 2.3, $[0, z]$ is an MV-effect algebra. Further, if there are nonzero elements $z_1, z_2 \in C(\widehat{E})$ with $z_1 \oplus z_2 = z'$ then $E \cong [0, z] \times [0, z_1] \times [0, z_2]$, where $[0, z_1]$ and $[0, z_2]$ have at least two blocks each, because otherwise z_1 or z_2 is under $z \wedge z' = 0$, a contradiction. We conclude that z' is an atom of $C(\widehat{E})$. It follows that $[0, z']$ has a trivial center, by Lemma 2.3. By Lemma 2.2, $[0, z']$ is a horizontal sum of its blocks D_1 and D_2. Let $a \in D_1 \setminus \{0, 1\}$ and $b \in D_2 \setminus \{0, 1\}$. As E is supremum-dense in $\widehat{E}$, there are nonzero elements $u \in D_1 \cap E$ and $v \in D_2 \cap E$ such that $u \leq a$ and $v \leq b$, which gives that $u \vee v = z' \in E$. Hence $z' \in C(E) = \{0, 1\}$ and so $z' = 0$, a contradiction. We conclude that $C(\widehat{E}) = \{0, 1\}$ and hence $\widehat{E}$ is a horizontal sum of $\widehat{M_1}$ and $\widehat{M_2}$. It follows that $\{0, 1\} \subseteq M_1 \cap M_2 \subseteq \widehat{M_1} \cap \widehat{M_2} = \{0, 1\}$, which proves the theorem.

Corollary 2.5. *On every Archimedean lattice effect algebra E with two blocks and trivial center there exists a state. If E is atomic then there exists an (o)-continuous state on E.*

Proof. By Theorem 2.4, for $x \in M_1$ and $y \in M_2$, where M_1, M_2 are blocks of E, we have $x \leq y'$ iff at least one of x and y is equal to zero. Because M_1 and M_2 are MV-effect algebras, i.e., can be organized into MV-algebras, there exist states ω_1 on M_1 and ω_2 on M_2. Let $\omega(x) = \omega_1(x)$ for all $x \in M_1$ and $\omega(x) = \omega_2(x)$ for all $x \in M_2$. Then ω is a state on E. If E is atomic then M_1 and M_2 are atomic and Archimedean. By [16] there are (o)-continuous states ω_1 on M_1 and ω_2 on M_2. Hence ω is (o)-continuous.

Note that a state ω in Corollary 2.5 need not be *subadditive*, i.e., $\omega(a \vee b) \leq \omega(a) + \omega(b)$ need not hold for all $a, b \in E$.

Example 2.6. Let $E = \{0, a, 2a, b, 1\}$ where $1 = 3a = 2b$. Hence E is a horizontal sum of two chains $\{0, a, 2a, 1 = 3a\}$ and $\{0, b, 1 = 2b\}$. Evidently there is the unique state ω on E for which $1 = \omega(a \vee b) \not\leq \omega(a) + \omega(b) = \frac{1}{3} + \frac{1}{2}$.

3. Decompositions of complete effect algebras

A *direct product* of a family $\{E_\varkappa \mid \varkappa \in H\}$ of effect algebras is the Cartesian product $\prod\{E_\varkappa \mid \varkappa \in H\}$ with "coordinatewise" defined operations, which means that $(a_\varkappa)_{\varkappa \in H} \oplus (b_\varkappa)_{\varkappa \in H} = (a_\varkappa \oplus_\varkappa b_\varkappa)_{\varkappa \in H}$ iff $a_\varkappa \oplus_\varkappa b_\varkappa$ is defined in $E_\varkappa$ for all $\varkappa \in H$. Further, $(0_\varkappa)_{\varkappa \in H}$ is the zero and $(1_\varkappa)_{\varkappa \in H}$ is the unit in the product.

Lemma 3.1 [17]. *Let $(E;\oplus,0,1)$ be a complete effect algebra and let $D\subseteq C(E)$. Let $\bigvee D=1$ and $d_1\wedge d_2=0$ for all $d_1\neq d_2$, $d_1,d_2\in D$. Then the effect algebra E is isomorphic to a direct product $\prod\{[0,d]\mid d\in D\}$.*

Theorem 3.2. *Let E be a complete atomic effect algebra with finitely many blocks and nontrivial center. Then*

(i) *$C(E)$ is a complete atomic Boolean algebra.*
(ii) *$E\cong\prod\{[0,p_\varkappa]\mid\varkappa\in H\}$, where $\{[p_\varkappa\mid\varkappa\in H\}$ is the set of all atoms of $C(E)$.*
(iii) *For every atom p of $C(E)$ the interval $[0,p]$ is a complete atomic effect algebra with trivial center.*
(iv) *Every block of E is isomorphic to a direct product $\prod\{B_\varkappa\mid\varkappa\in H\}$ for some blocks $B_\varkappa$ of $[0,p_\varkappa]$, and conversely.*
(v) *The number n of all blocks of E is equal to the number of all different possibilities of products $\prod\{B_\varkappa\mid\varkappa\in H\}$, for all blocks $B_\varkappa$ of $[0,p_\varkappa]$, $\varkappa\in H$.*
(vi) *If E is not an MV-effect algebra then there are atoms $p_1,p_2,\dots,p_k$ of $C(E)$ such that E is isomorphic to*
$$M\times[0,p_1]\times\cdots\times[0,p_k]$$
where M is a complete atomic MV-effect algebra or $M=\{0\}$ and $[0,p_i]$ for $i=1,\dots,k$, are irreducible complete atomic effect algebras with at least two blocks each.

Proof. (i) For the proof that $C(E)$ is a Boolean algebra we refer the reader to [5]. By [15] $C(E)$ is complete. Let $z\in C(E)$ and a is an atom of E such that $a\leq z$. Then $w=\bigwedge\{y\in C(E)\mid a\leq y\}\in C(E)$ and $w\leq z$. Let $v\in C(E)$, $v\neq 0$ and $v<w$. Then $a\not\leq v$ and hence $a\leq v'$ which gives $v\leq w\leq v'$, a contradiction. Thus w is an atom of $C(E)$.

(ii) If $\{p_\varkappa\mid\varkappa\in H\}$ is the set of all atoms of $C(E)$ then evidently $\bigvee\{p_\varkappa\mid\varkappa\in H\}=1$ and $p_{\varkappa_1}\wedge p_{\varkappa_2}=0$ for all $\varkappa_1\neq\varkappa_2$. Thus the statement follows by Lemma 3.1.

(iii) follows by Lemma 2.3.

(iv) Clearly, $(a_\varkappa)_{\varkappa\in H}\leftrightarrow(b_\varkappa)_{\varkappa\in H}$ iff $a_\varkappa\leftrightarrow b_\varkappa$ for all $\varkappa\in H$ because the operations $\oplus$, $\vee$ and $\wedge$ in the product are defined coordinatewise. Hence (iv) follows by maximality of blocks.

(v) is a consequence of (iv).

(vi) The effect algebra $[0,p_\varkappa]$ has a unique block iff it is an MV-effect algebra. Let $H_1=\{\varkappa\in H\mid[0,p_\varkappa]$ has a unique block$\}$. Then $M=\prod\{[0,p_\varkappa]\mid\varkappa\in H_1\}$ is an MV-effect algebra, which is evidently complete and atomic. If E is not an MV-effect algebra then for some atoms $p_\varkappa$ of $C(E)$ the effect algebra $[0,p_\varkappa]$ has at least two blocks. Evidently, there are only finitely many such atoms $p_\varkappa$ because E has only finitely many blocks.

Theorem 3.3. *Let E be a complete effect algebra with exactly n blocks and nontrivial center. If $n>1$ then:*

(i) *$C(E)$ has at least one atom.*
(ii) *If E is not an MV-effect algebra then there are atoms $p_1,\dots,p_k$ of $C(E)$ such that $E\cong M\times[0,p_1]\times\cdots\times[0,p_k]$ where M is a complete MV-effect algebra or $M=\{0\}$ and $[0,p_1],\dots,[0,p_k]$ are irreducible complete effect algebras with at least two blocks each.*

(iii) *If n is a prime number, then there is an atom p of $C(E)$ such that $E \cong M \times [0,p]$ where $M \neq \{0\}$ is a complete MV-effect algebra and $[0,p]$ is an irreducible effect algebra with exactly n blocks.*

Proof. (i), (ii): Let $A = \{d \in C(E) \mid d \neq 0$ and $[0,d]$ is an MV-effect algebra$\}$. If $A = \emptyset$, we put $M = \{0\}$ and $w = 0$. If $A \neq \emptyset$, we put $M = [0,w]$. By Lemma 2.3, $[0,w]$ is an MV-effect algebra. Further, for every nonzero $d \in C(E)$ with $d \leq w'$ the effect algebra $[0,d]$ has at least two blocks. Otherwise, we have $d \leq w \wedge w' = 0$, a contradiction. Thus there is only finite set of nonzero elements $d_1, d_2, \dots, d_m \in C(E)$ such that $w' = d_1 \oplus d_2 \oplus \cdots \oplus d_m$, because $[0,d_1], \dots [0,d_m]$ has at least two blocks each and $[0,w']$ has exactly n blocks under which $[0,w'] \cong [0,d_1] \times \cdots \times [0,d_m]$. We conclude that there are atoms $p_1, \dots, p_k$ of $C(E)$ such that $[0,w'] \cong [0,p_1] \times \cdots \times [0,p_k]$ and hence $E \cong M \times [0,p_1] \times \cdots \times [0,p_k]$ where $[0,p_1], \dots, [0,p_k]$ are irreducible with at least two blocks each and M is a complete MV-effect algebra or $M = \{0\}$. Since E is not an MV-effect algebra, (i) is also proved.

(iii) If n is a prime number then by (ii) there is an atom p of $C(E)$ such that $E \cong M \times [0,p]$. Since $C(E) \neq \{0,1\}$ we conclude that $p \neq 1$ and hence $M \neq \{0\}$.

4. The existence of (order-continuous) subadditive states

Example 2.3 shows that a state on a lattice effect algebra need not be subadditive. On the other hand, it was proved in [16] that on every Archimedean atomic distributive effect algebra there exists an order-continuous subadditive state (a probability). Note that MV-effect algebras are distributive effect algebras. Finally, note that a state ω on a lattice effect algebra E is subadditive iff $\omega(a \vee b) \leq \omega(a) + \omega(b)$ for all $a, b \in E$ iff $\omega(a \vee b) = \omega(a) + \omega(b)$ for all $a, b \in E$ with $a \wedge b = 0$ iff $\omega(a) + \omega(b) = \omega(a \vee b) + \omega(a \wedge b)$ for all $a, b \in E$ iff ω is a *valuation*, [14]

Theorem 4.1. *Let E be a complete effect algebra with exactly n blocks and nontrivial center.*

(i) *If n is a prime number then there exists a subadditive state on E.*

(ii) *If E is atomic and n is a prime number then there exists an (o)-continuous subadditive state on E (a probability).*

(iii) *If $n = 2k$ and k is a prime number then there exists a state on E. If, moreover, E is atomic then there exists an (o)-continuous state on E.*

Proof. (i) and (ii): If $n = 1$ the proof follows by [16]. Let $n > 1$. By Theorem 3.3, $E \cong M \times [0,p]$ where $M \neq \{0\}$ is a complete MV-effect algebra (can be organized into an MV-algebra) hence there is a subadditive state ω_1 on M. Thus $\omega \colon E \to [0,1] \subseteq R$ defined by $\omega\big((x,y)\big) = \omega_1(x)$ for all $(x,y) \in M \times [0,p]$ is a subadditive state on E. Moreover, if E is atomic then by Theorem 3.2, M is a complete atomic MV-effect algebra. By [16] there is an (o)-continuous subadditive state ω_1 on M. Hence the state ω defined above is also (o)-continuous and subadditive.

(iii) If $n = 2k$, k is a prime number and E is atomic then by Theorem 3.3 either $E \cong M \times [0,p]$ or $E \cong M \times [0,p_1] \times [0,p_2]$, where M is a complete MV-effect algebra $[0,p]$, $[0,p_1]$, $[0,p_2]$ are complete irreducible effect algebras under which $[0,p]$ has exactly n blocks, $[0,p_1]$ has two blocks and $[0,p_2]$ has k blocks. In the first case $M \neq \{0\}$. By Corollary 2.5, there exists a state on $[0,p_1]$. Since a state on M exists, we conclude that there exists a state on E. If E is atomic then by [16] and Corollary 2.5, all these states can be (o)-continuous.

Remark 4.2. A lattice effect algebra E with finitely many blocks can be supremum-densely embedded (as a sub-effect algebra and a full sub-lattice) into a complete effect algebra $\widehat{E}$ if and only if E is Archimedean, [20]. Moreover, if M_k, $k = 1, \dots, n$, are blocks of E which cover E, that means $E = \bigcup_{k=1}^n M_k$, then $\widehat{E} = \bigcup_{k=1}^n \widehat{M_k}$ where $\widehat{M_k}$ are blocks of $\widehat{E}$ such that $M_k \subseteq \widehat{M_k}$, $k = 1, 2, \dots, n$ (see [20], Theorem 4.3). Here, $\{M_1, \dots, M_k\}$ need not be the set of all blocks of E (see [9]). Conversely, if $\widehat{E} = \bigcup_{k=1}^n \widehat{M_k}$ then there are blocks M_k of E such that $\widehat{M_k} \cap E \subseteq M_k$ and hence $E = \bigcup_{k=1}^n M_k$. It follows that the minimal number n_0 of blocks of E which cover E is equal to the minimal number of blocks of $\widehat{E}$ which cover $\widehat{E}$.

Theorem 4.3. *Let E be an Archimedean lattice effect algebra with finitely many blocks and nontrivial center. Let n_0 be the minimal number of blocks which cover E. If n_0 is a prime number then there is a subadditive state on E. If, moreover, E is atomic then there is an (o)-continuous and subadditive state on E.*

Proof. Let $\widehat{E}$ be a complete effect algebra in which E is (up to isomorphism) a supremum-dense sub-effect algebra, [20]. Then $C(E) \subseteq C(\widehat{E})$ and hence $C(\widehat{E}) \neq \{0, 1\}$. Further, n_0 is a minimal number of blocks of $\widehat{E}$ which cover $\widehat{E}$. Assume $n_0 > 1$. Let $w = \bigvee\{d \in C(\widehat{E}) \mid [0, d]_{\widehat{E}}$ has a unique block$\}$. By Lemma 2.3, $w \in C(\widehat{E})$, $[0, w]_{\widehat{E}}$ is an MV-effect algebra. Moreover, for every nonzero $w_1 < w'$ the effect algebra $[0, w_1]_{\widehat{E}}$ has at least two blocks. Further, $\widehat{E} \cong [0, w]_{\widehat{E}} \times [0, w']_{\widehat{E}}$ which implies that the minimal number of blocks of $[0, w']_{\widehat{E}}$ which cover $[0, w']_{\widehat{E}}$ is n_0. It follows that w' is an atom of $C(\widehat{E})$. Really, if there is a nonzero element $w_1 \in C(\widehat{E})$ with $w_1 < w'$ then $w' = w_1 \oplus (w' \ominus w_1)$ and $[0, w']_{\widehat{E}} \cong [0, w_1]_{\widehat{E}} \times [0, w' \ominus w_1]_{\widehat{E}}$ cannot be covered by n_0 blocks of $[0, w']_{\widehat{E}}$ as n_0 is a prime number and every $[0, w_1]_{\widehat{E}}$ and $[0, w' \ominus w_1]_{\widehat{E}}$ has at least two blocks each. Since $C(\widehat{E}) \neq \{0, 1\}$, we obtain that $w' \neq 1$ and hence $w \neq 0$ which gives $M = [0, w]_{\widehat{E}} \neq \{0\}$.

Now, the existence of a subadditive state on M implies the existence of a subadditive state $\widehat{\omega}$ on $\widehat{E}$ and the restriction $\widehat{\omega}|_E$ is a subadditive state on E. If E is atomic then $\widehat{E}$ and M are atomic and $\widehat{\omega}$ can be assumed (o)-continuous by [16].

Corollary 4.4. *Let E be a finite lattice effect algebra with nontrivial center. If the number n of all blocks of E, or the minimal number n_0 of blocks which cover E, is a prime number then there exists a subadditive state on E. If $n = 2k$ and k is a prime number then there exists a state on E.*

Remark 4.5. Since every orthomodular lattice L can be organized into an Archimedean lattice effect algebra by putting $a \oplus b = a \vee b$ for all orthogonal pairs $a, b \in L$, we can adopt results of Theorems 4.1 and 4.3 and Corollary 4.4 for orthomodular lattices (see [9] and references given there).

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