On the Generalized von Kármán System for Viscoelastic Plates. I. Pseudoparabolic Model

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Abstract

We deal with the system of quasistationary von Kármán equations describing moderately large deflections of thin viscoelastic plates. We shall concentrate on a differential-type material, which gives rise to a quasistationary system with a linear pseudoparabolic main part and a nonlinear differential term. This model arises when considering a special relaxation function involving only one exponential function. The existence and the uniqueness of a solution as the limit of a semidiscrete approximation is verified. Its behaviour for large values of the time variable is studied.

Key words: von Kármán equations, viscoelastic plate, pseuspoparabolic problem, semidiscretization, existence, uniqueness

MSC (2000): 74D10, 74K20, 65N40, 35K70

1 Introduction

Theodor von Kármán [10] stated the nonlinear system of partial differential equations for great deflections and for the Airy stress function of a thin elastic plate. This system has been treated systematically since the sixties by Berger and Fife [1] who have proved the existence of buckled states for a plate subjected only to compressive forces. A global existence theorem for a plate acted upon perpendicular and lateral loading has been established by Knightly [12]. Ciarlet [6] justified the von Kármán system as the plate model derived from the equations for a 3-dimensional body. In a comprehensive work, Fox, Raoult and Simo [8] showed that the von Kármán model, or more generally Kirchhoff-Love model, actually arises as the third of a hierarchy of plate models when the orders of magnitude of loads decrease, the first two models being the membrane model and the bending model. They give as well the orders of magnitude of the displacements, thus giving firm grounds to the assertion that the von Kármán model accounts for moderately large displacements.

The von Kármán system for viscoelastic plates was derived by J.Brilla [4], who considered the linearized stability problem for the generalized n-th order viscoelasticity. We considered in [2] the short memory anisotropic case, where we derived and solved the pseudoparabolic canonical equation with zero initial conditions and a nonlinear integral term. The integral term has in the anisotropic case a rather complicated form defined by the matrix exponential function.

In the present paper the memory term appears also in the equation for the Airy stress function. We deal with the long memory isotropic case; where the nonlinear system and the corresponding canonical equation can be derived in the same way as the traditional elastic von Kármán system. In a special case of the exponential relaxation function the integro-differential stress-strain relation can be converted into a differential relation with a nonzero initial condition. The main part of resulting system of equations is linear and pseudoparabolic and a nonlinear differential expression arises in the right-hand side of the equation for the Airy stress function. The nonlinear pseudoparabolic character remains also in the canonical equation acting in the Sobolev space of admissible deflections. The presented model is restricted to slow, quasistatic relaxation behaviour only and we neglect inertia forces. We derive conditions for the existence and uniqueness of a solution of the canonical initial value problem using the Rothe's method with respect to the time variable in a way similar to [9], where the parabolic problem was considered. Convergence is obtained when an integro-differential expression depending on the data remains bounded. This condition implies also the uniqueness. When a more restrictive condition is satisfied, we are able to give the behaviour of the solution for large values of the time variable.

2 Formulation of the Problem

We consider a thin isotropic plate occupying the domain

$$Q = \{(x, z) \in R^3;\ x = (x_1, x_2) \in \Omega,\ -h/2 < z < h/2\},$$

where Ω is a bounded simply connected domain in R^2 with a Lipschitz boundary Γ. The plate is clamped on its boundary and subjected to a perpendicular load $f(t, x),\ t > 0,\ x \in \Omega$. We restrict here to most simple boundary conditions. More general and nonhomogeneous boundary conditions can be considered too. We shall deal with them in the second part of the paper, which will be devoted to the general long memory material ([3]).

Assuming moderately large deflections and considering Kirchhoff-Love hypothesis ([16]) we have the strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i w \partial_j w) - z\partial_{ij} w,\ i,j = 1,2, \quad \varepsilon_{13} = \varepsilon_{23} = 0.$$

Let $\{\sigma^{ij}\}$ be the stress tensor fulfilling the condition $\sigma^{33} = 0$. The principle of virtual displacements holds in the form

$$\int_\Omega \left(\int_{-h/2}^{h/2} \sigma^{ij} \delta\varepsilon_{ij} dz \right) dx = \int_\Omega f(t,x) v(x) dx \text{ for all } (\omega_1, \omega_2, v) \in U \times U \times V,$$

where v and ω_i are virtual displacements in the directions z and x_i $(i = 1,2)$ respectively and $U = H_0^1(\Omega),\ V = H_0^2(\Omega)$ are the spaces of admissible displacements. The virtual strains are of the form

$$\delta\varepsilon_{ij} = \frac{1}{2}(\partial_i \omega_j + \partial_j \omega_i + \partial_i w \partial_j v + \partial_i v \partial_j w) - z\partial_{ij} v,\ i,j = 1,2.$$

The principle of virtual displacements implies that the stress resultants

$$N_{ij} = \int_{-h/2}^{h/2} \sigma^{ij} dz$$

satisfy the homogeneous equations $\partial_j N_{ij} = 0, \quad i,j = 1,2.$

Then there exists the Airy stress function $\Phi : \Omega \to R$ defined by the equations

$$N_{11} = \partial_{22}\Phi, \quad N_{22} = \partial_{11}\Phi, \quad N_{12} = -\partial_{12}\Phi.$$

The stress-strain relations for isotropic viscoelastic long memory materials of Boltzmann type are of the form

$$\begin{gathered} \sigma^{ij} = \tfrac{E(0)}{1-\mu^2}[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}] + \tfrac{E'}{1-\mu^2} * [(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}](t), \quad (1) \\ \delta_{11} = \delta_{22} = 1,\ \delta_{12} = \delta_{21} = 0, \quad \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22}, \quad \sigma^{33} = 0 \end{gathered}$$

with a Poisson ratio $\mu \in (0, \frac{1}{2})$, a positive decreasing relaxation function $E \in C^1(R^+)$ and a convolution product

$$(f * g)(t) = \int_0^t f(t-s) g(s) ds.$$

Let us set

$$[v, w] = \partial_{11}v\partial_{22}w + \partial_{22}v\partial_{11}w - 2\partial_{12}v\partial_{12}w, \quad v,\ w \in H^2(\Omega). \tag{2}$$

We recall that in the elastic case the well known von Kármán system for the deflection w and the Airy stress function Φ has the form ([7])

$$D_0\Delta^2 w - [\Phi, w] = f(x) \text{ in } \Omega, \quad w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3}$$

$$\Delta^2\Phi = -\frac{E_0 h}{2}[w, w] \text{ in } \Omega, \quad \Phi = \frac{\partial \Phi}{\partial \nu} = 0 \text{ on } \Gamma, \tag{4}$$

$$E_0 > 0,\ D_0 = \frac{h^3 E_0}{12(1-\mu^2)}.$$

In order to convert a system (3), (4) into one equation for a deflection function w it is classical to introduce the bilinear operator $B : V \times V \to V$ defined by the uniquelly solved equation

$$((B(u, v), \varphi)) \ = \ \int_\Omega [u, v]\varphi dx \text{ for all } \varphi \in V, \tag{5}$$

where

$$((u, v)) = \int_\Omega \Delta u \Delta v dx, \quad \|u\| = ((u, u))^{1/2}$$

are the inner product and the norm in the Sobolev space $V = H_0^2(\Omega)$.

The function $y \equiv B(u, v) \in V$ is a weak solution of the boundary value problem

$$\Delta^2 y = [u, v] \text{ in } \Omega, \quad y = \frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma.$$

The existence and the uniqueness of $B(u, v)$ is a consequence of the fact that the form $(v, w) \to [u, v]$ represents the bilinear continuous mapping $[.,] : H^2(\Omega) \times H^2(\Omega) \to L^1(\Omega)$ and we have the compact imbedding $L^1(\Omega) \subset H^{-2}(\Omega) = V^*$ - the dual space of V. Using the Green formula the following formula for the trilinear form can be derived (see [7])

$$\int_\Omega [u, v]w dx = \int_\Omega u[v, w]dx \text{ for all } u,\ v,\ w\ \in V. \tag{6}$$

Moreover there exists a constant c such that

$$|\int_\Omega [u, v]w dx| \le c\|u\|\|v\|_{1,4}\|w\|_{1,4} \text{ for all } u, v, w \in V,$$

where $\|.\|_{1,4}$ is a norm in the Sobolev space $W^{1,4}(\Omega)$. The operator B then posseses the property

$$((B(B(u,v),w),y)) = ((B(u,v), B(w,y))) \text{ for all } u,v,w,y \in V. \qquad (7)$$

Moreover it is compact and symmetric.

Expressing a weak solution of the boundary value problem (4) by (5) and inserting it into the equation (3) we obtain the nonlinear boundary value problem

$$D_0\Delta^2 w + \frac{hE_0}{2}[B(w,w),w] = f(x) \text{ in } \Omega, \quad w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma. \qquad (8)$$

Let us define the element $q \in V$ uniquely defined as a solution of the identity

$$((q,\varphi)) = \frac{1}{D_0}\int_\Omega f\varphi dx \text{ for all } \varphi \in V \qquad (9)$$

and the operator

$$C : V \to V, \quad C(v) = \alpha B(B(v,v),v), \quad \alpha = \frac{hE_0}{2D_0} = \frac{6(1-\mu^2)}{h^2}. \qquad (10)$$

We formulate the operator equation in the space V

$$w + C(w) = q, \quad w \in V. \qquad (11)$$

We can consider a solution of the equation (11) as a weak solution of the nonlinear problem (8). The equation (11) is called the canonical equation for the boundary value problem (3), (4).

The operator $C : V \to V$ is compact and not negative. It holds

$$((C(v),v)) = \alpha\|B(v,v)\|^2, \quad v \in V. \qquad (12)$$

Moreover C fulfils ([7]) the inequality

$$((C(u) - C(v), u - v)) \leq \alpha\|B\|^2_{L(V\times V;V)} \max\{\|u\|^2, \|v\|^2\}\|u - v\|^2 \qquad (13)$$

which is very important in the continuity and uniqueness considerations.

The following existence and uniqueness theorem follows.

Theorem 2.1 *For every $q \in V$ there exists a solution $w \in V$ of the canonical equation (11). If*

$$\|q\| < \frac{1}{\sqrt{\alpha}\|B\|}, \qquad (14)$$

then a solution of (11) is unique.

Proof. The compactness and symmetry of the operator B together with (12) imply that there exists a solution $w \in V$ of (11) which is simultaneously a solution of the minimum problem

$$J(w) = \min_{v \in V} J(v), \;\; J(v) = \frac{1}{2}\|v\|^2 + \frac{\alpha}{4}\|B(v,v)\|^2 - ((q,v)).$$

If $w_1, \; w_2 \; \in V$ are two solution of (11), then the difference $w = w_2 - w_1$ satisfies the relation

$$w = C(w_1) - C(w_2).$$

The inequality (13) implies

$$\|w\|^2 \leq \alpha\|B\|^2 \max\{\|w_1\|^2, \|w_2\|^2\}\|w\|^2 \tag{15}$$

If the bound (14) holds, then applying the relation (12) we obtain

$$\|w_i\|^2 \leq \|q\|^2 < \frac{1}{\alpha\|B\|^2}, \;\; i = 1,2$$

and the uniqueness is the consequence of (15). $\square$

Let us define the material function $D(t) = \frac{h^3}{12(1-\mu^2)}E(t)$. Applying the principle of virtual displacements and the viscoelastic stress-strain relations (1) the following integro-differential von Kármán system for the deflection $w(t,x)$ and the Airy stress function $\Phi(t,x), \; t \geq 0, \; x \in \Omega$ can be derived:

$$D(0)\Delta^2 w + D' * \Delta^2 w - [\Phi, w] = f(t), \;\; w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma, \tag{16}$$

$$\Delta^2 \Phi = -\frac{h}{2}(E(0)[w,w] + E' * [w,w]), \quad \Phi = \frac{\partial \Phi}{\partial \nu} = 0 \text{ on } \Gamma. \tag{17}$$

3 Zener model. Existence and Uniqueness

Most of long memory viscoelastic material are modelled by the relaxation function of the form ([5])

$$E(t) = E_0 + \sum_{i=1}^{k} e^{-\beta_i t}, \;\; E_0 > 0, \;\; E_i > 0, \;\; \beta_i > 0, \;\; i = 1, ..., k.$$

It is possible in the case $k = 1$ to transform the integro-differential stress-strain relations into first order differential relations. Actually, let

$$E(t) = E_0 + E_1 e^{-\beta t}, \;\; E_0 > 0, \;\; E_1 > 0, \;\; \beta > 0.$$

After differentiating relation (1) we obtain the following differential stres-strain relation with the initial conditions characterizing the Zener viscoelastic model

$$\sigma'_{ij} + \beta\sigma_{ij} =$$
$$\frac{E_0 + E_1}{1-\mu^2}[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}]' + \frac{\beta E_0}{1-\mu^2}[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}], \quad (18)$$
$$\sigma_{ij}(0) = \frac{E_0 + E_1}{1-\mu^2}[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}](0). \quad (19)$$

Let us define the material constants $D_i = \frac{h^3}{12(1-\mu^2)}E_i,\ i = 0, 1.$ Applying the same approach as in the elastic case the following pseudoparabolic von Kármán system for the deflection w and the Airy stress function Φ can be derived:

$$(D_0 + D_1)\Delta^2 w' + \beta D_0\Delta^2 w - [\Phi, w] = f'(t) + \beta f(t),\ x \in \Omega,$$
$$w = \frac{\partial w}{\partial\nu} = 0 \text{ on } \Gamma, \quad (20)$$
$$\Delta^2\Phi = -\frac{h}{2}\left((E_0 + E_1)[w, w]' + \beta E_0[w, w]\right),\ x \in \Omega,$$
$$\Phi = \frac{\partial\Phi}{\partial\nu} = 0 \text{ on } \Gamma, \quad (21)$$
$$w(0, x) = w_0(x),\ x \in \Omega, \quad (22)$$

where the initial deflection fulfils the stationary von Kármán system

$$(D_0 + D_1)\Delta^2 w_0 - [\Phi_0, w_0] = f(0), \quad w_0 = \tfrac{\partial w_0}{\partial\nu} = 0 \text{ on } \Gamma, \quad (23)$$
$$\Delta^2\Phi_0 = -\tfrac{h}{2}(E_0 + E_1)[w_0, w_0], \quad \Phi_0 = \tfrac{\partial\Phi_0}{\partial\nu} = 0 \text{ on } \Gamma. \quad (24)$$

Using the definition (5) of the bilinear operator $B : V \times V \to V$ we arrive at the nonlinear pseudoparabolic initial-boundary value problem for the deflection w :

$$(D_0 + D_1)\Delta^2 w' + \beta D_0\Delta^2 w + \frac{h}{2}[(E_0 + E_1)B(w, w)' + \beta E_0 B(w, w), w]$$
$$= f'(t) + \beta f(t),\ x \in \Omega, \quad (25)$$
$$w(t) = \frac{\partial w(t)}{\partial\nu} = 0 \text{ on } \Gamma,\ t \geq 0, \quad (26)$$
$$(D_0 + D_1)\Delta^2 w(0) + \frac{h}{2}[(E_0 + E_1)B(w(0), w(0)), w(0)] = f(0). \quad (27)$$

A weak formulation of the problem can be expressed as a nonlinear pseudoparabolic initial value problem in the Hilbert space V:

$$w'(t) + aw(t) + bB(B(w,w)' + aB(w,w),w)(t) = q'(t) + \beta q(t), \quad (28)$$
$$w(0) + bB(B(w(0),w(0)),w(0)) = q(0), \quad (29)$$

where

$$a = \frac{\beta D_0}{D_0 + D_1} = \frac{\beta E_0}{E_0 + E_1}, \quad b = \frac{h(E_0 + E_1)}{2(D_0 + D_1)} = \frac{6(1-\mu^2)}{h^2}$$

and the function $q : [0,\infty) \to V$ is uniquely defined as a solution of the identity

$$((q(t),v)) = \frac{1}{D_0 + D_1}\langle f(t),v\rangle \text{ for all } v \in V.$$

Definition 3.1 *Let $f \in C^1([0,\infty),V^*)$. If $w : [0,\infty) \to V$ is a solution of the initial value problem (28), (29) and*

$$\Phi(t) = -\frac{h}{2}[(E_0 + E_1)B(w,w)' + \beta E_0 B(w,w)], \ t > 0,$$

then a function $t \to \{w(t),\Phi(t)\}$ is a weak solution of the initial-boundary value problem (20)-(24).

We now turn to solving the initial value problem (28), (29) by the semidiscretization or Rothe's method with respect to the time variable. We convert problem (28), (29) into a sequence of stationary von Kármán problems .

For a fixed integer N we set

$$\tau = \tfrac{T}{N}, \ t_i = i\tau, \ w_i = w(t_i), \ i = 0,1,...,N;$$
$$\delta w_j = \tfrac{1}{\tau}(w_j - w_{j-1}), \ j = 1,...,N.$$

Applying the discrete values w_i and the finite differences δw_i in (28), (29) we obtain the nonlinear equations in the space V:

$$w_0 + bB(B(w_0,w_0),w_0) = q_0, \quad (30)$$
$$\delta w_i + aw_i + bB(\delta B(w_i,w_i) + aB(w_i,w_i),w_i)$$
$$= \delta q_i + \beta q_i, \ i = 1,...N. \quad (31)$$

Both equations above have solutions w_0 and $w_i, \ i = 1,...,N$ respectively. They are minimizers of the problems

$$J_i(w_i) = \min_{v \in V} J_i(v), \ i = 0,1,...,N, \quad (32)$$

where

$$J_0(v) = \frac{1}{2}\|v\|^2 + \frac{b}{4}\|B(v,v)\|^2 - ((q_0, v)), \tag{33}$$

$$\begin{aligned} J_i(v) = {} & \tfrac{1}{2}(1+\tau a)[\|v\|^2 + \tfrac{b}{2}\|B(v,v)\|^2] - \tfrac{b}{2}((B(w_{i-1}, w_{i-1}), B(v,v))) \\ & -((w_{i-1} + \tau(\delta q_i + \beta q_i), v)), \quad i = 1, ...N. \end{aligned} \tag{34}$$

We continue with deriving the a priori estimates of w_i and δw_i.

Lemma 3.2 *Let $\alpha < \frac{a}{2}$. There exists $\tau_0 > 0$ such that*

$$\|w_i\|^2 \le e^{-2\alpha\tau i}\|q_0\|^2 + \frac{1}{a}\sum_{j=1}^{i} \tau e^{-\alpha\tau(1+2i-2j)}\|\delta q_j + \beta q_j\|^2, \tag{35}$$

$i = 1, ..., N,\ 0 < \tau \le \tau_0$.

Proof. In order to obtain a priori estimates not depending on the length T of the time interval we express the values w_i in a form

$$w_i = e^{-\alpha\tau i}u_i,\ \alpha > 0,\ i = 0, 1, ..., N. \tag{36}$$

We have the following expression of the difference δw_i :

$$\delta w_i = (\delta e^{-\alpha\tau i})u_i + e^{-\alpha\tau(i-1)}\delta u_i,\ i = 1, ..., N. \tag{37}$$

After setting $i = j$ in (31) and multiplying with $e^{\alpha\tau j}u_j$ in the Hilbert space V we obtain the identity

$$\begin{gathered} e^{\alpha\tau}((\delta u_j, u_j)) + (a - \tfrac{e^{\alpha\tau}-1}{\tau})\|u_j\|^2 + be^{-2\alpha\tau(j-1)}((\delta B(u_j, u_j), B(u_j, u_j))) \\ + be^{-2\alpha\tau j}(a - \tfrac{e^{2\alpha\tau}-1}{\tau})\|B(u_j, u_j)\|^2 = e^{\alpha\tau j}((\delta q_j + \beta q_j, u_j)), \\ j = 1, ..., n. \end{gathered}$$

Summing up and using the relations

$$2\tau\sum_{j=1}^{i}((\delta u_j, u_j)) = \|u_i\|^2 - \|u_0\|^2 + \sum_{j=1}^{i}\tau^2\|\delta u_j\|^2,$$

$$\sum_{j=1}^{i} e^{\alpha\tau}e^{-2\alpha\tau j}((\ \delta B(u_j, u_j), B(u_j, u_j)\ )) =$$

$$\sum_{j=1}^{i}((\delta(e^{-\alpha\tau j}B(u_j, u_j)\ ), e^{-\alpha\tau j}B(u_j, u_j)\ )) + \frac{e^{\alpha\tau}-1}{\tau}\sum_{j=1}^{i} e^{-2\alpha\tau j}\|B(u_j, u_j)\|^2$$

we arrive at the inequality

$$e^{\alpha\tau}(\|u_i\|^2 + be^{-2i\alpha\tau}\|B(u_i,u_i)\|^2)+$$

$$2(a-\alpha e^{\alpha\tau})\sum_{j=1}^{i}\tau\|u_j\|^2 + 2b(a-2\alpha e^{2\alpha\tau})\sum_{j=1}^{i}\tau e^{-2\alpha\tau j}\|B(u_j,u_j)\|^2$$

$$\leq\ e^{\alpha\tau}(\|u_0\|^2 + b\|B(u_0,u_0)\|^2) + 2\sum_{j=1}^{i}\tau e^{\alpha\tau j}((\delta q_j + \beta q_j, u_j)),$$

$$i = 1, ..., N.$$

We obtain directly from the equation (30) the estimate

$$\|w_0\|^2 + b\|B(w_0,w_0)\|^2 \leq \|q_0\|^2. \tag{38}$$

Setting $\tau_0 > 0$ such that

$$\alpha e^{\alpha\tau} \leq \frac{a}{2} \text{ for all } \tau \in (0,\tau_0) \tag{39}$$

we obtain considering $w_0 = u_0$ the estimate

$$\|u_i\|^2 \leq \|q_0\|^2 + \frac{1}{a}\sum_{j=1}^{i}\tau e^{\alpha\tau(2j-1)}\|\delta q_j + \beta q_j\|^2.$$

and the estimate (35) follows after using expression (36). $\square$

In order to obtain the uniform estimates of the differences we add the restriction on the bounds of the right-hand sides of the equations (28), (29).

Lemma 3.3 *Let $q \in C^1([0,T],V)$, $\alpha < \frac{a}{2}$. If*

$$e^{-2\alpha t}\|q(0)\|^2 + \frac{1}{a}\int_0^t e^{-2\alpha(t-s)}\|q'(s) + \beta q(s)\|^2 ds < \frac{1}{b\|B\|^2},\ t \in [0,T], \tag{40}$$

then there exists a constant C_2 not depending on $\tau > 0$ such that

$$\|\delta w_j\| \leq C_2.\ i = 1, ..., N. \tag{41}$$

Proof. After multiplying the equation (31) with δw_j in the space V we obtain

$$\|\delta w_i\|^2 + a((w_i,\delta w_i)) + \frac{1}{2}b\|\delta B(w_i,w_i)\|^2\ +$$
$$\frac{1}{2}b\tau((\delta B(w_i,w_i), B(\delta w_i,\delta w_i)) + ab((\ B(w_i,w_i), B(w_i,\delta w_i)\ ))$$
$$= ((\delta q_i + \beta q_i),\delta w_i)), \tag{42}$$

where we have used the relation

$$2B(w_i, \delta w_i) = \delta B(w_i, w_i) + \tau B(\delta w_i, \delta w_i).$$

The a priori estimate (35) and identity (42) further imply the inequality

$$\|\delta w_i\|^2 \leq C_1 + \frac{1}{4} b \|B(w_i - w_{i-1}, \delta w_i)\|^2, \tag{43}$$

where the constant C_1 depends only on the constants $a,\ b$ and the function q and its derivative.

Let us assume that

$$\|w_i\| < \frac{1}{b\|B\|^2},\ i = 1, ..., N. \tag{44}$$

Comparing with the a priori estimate (35) we can see that for sufficiently small $\tau > 0$ the condition

$$e^{-2\alpha\tau i}\|q_0\|^2 + \frac{1}{a}\sum_{j=1}^{i} \tau e^{-2\alpha\tau(i-j+1)}\|\delta q_j + \beta q_j\|^2 < \frac{1}{b\|B\|^2},\ i = 1, ..., N \tag{45}$$

is sufficient for estimate (44) to be satisfied.

Assuming $q \in C^1([0,T], V)$ we obtain $\tau_0 > 0$ such that for $\tau < \tau_0$ the bounds (44) hold.

Comparing with (42), (44) we obtain from (40) the a priori estimate (41) which completes the proof. $\Box$.

We can now formulate the existence and uniqueness theorem.

Theorem 3.4 *Let $q \in C^1([0,T]; V)$ be such that condition (40) is satisfied with $\alpha < \frac{a}{2}$. Then there exists a unique solution $w \in W^{1,\infty}(0,T;V)$ of the initial value problem (28), (29).*

There exists a subsequence of a sequence $\{w_n\}$ of segment line functions defined by discrete values w_i fulfilling the equations (30), (31) such that a weak-star convergence (47) holds.

Proof. Let us set for $n = 1, 2, ...$

$$N \equiv N(n),\ \tau \equiv \tau_n = \frac{T}{N(n)},\ \lim_{n\to\infty} N(n) = \infty.$$

We define the following functions determined by discrete values $w_i \equiv w_i^n,\ \delta w_i \equiv \delta w_i^n$:

$$\begin{aligned}
&w_n : [0,T] \to V,\ w_n(t) = w_{i-1}^n + (t - t_{i-1}^n)\delta w_i^n,\ t_{i-1}^n \leq t \leq t_i^n,\\
&\bar{w}_n : [0,T] \to V,\ \bar{w}_n(0) = w_0,\ \bar{w}_n(t) = w_i,\ t_{i-1}^n < t \leq t_i^n,\\
&t_i^n = i\tau_n,\ i = 0, 1, ..., N(n).
\end{aligned}$$

From the previous a priori estimates we know that the sequence of functions $\{w_n\}$ is bounded in the Sobolev space $W^{1,\infty}(0,T;V)$:

$$\|w_n\|_{W^{1,\infty}(0,T;V)} \leq C_3,\ n = 1,2,\dots\ . \tag{46}$$

Then there exists a subsequence (again denoted by $\{w_n\}$) and a function $w \in W^{1,\infty}(0,T;V)$ such that

$$w_n \rightharpoonup^* w \qquad \text{in} \qquad W^{1,\infty}(0,T;V), \tag{47}$$

$$w_n(t) \rightharpoonup w(t),\ \bar{w}_n(t) \rightharpoonup w(t) \text{ in } V \ \text{ for every } t \in [0,T], \tag{48}$$

$$w_n \rightharpoonup^* w,\ \bar{w}_n \rightharpoonup^* w,\ w_n' \rightharpoonup^* w' \text{ in } L^\infty(0,T;V), \tag{49}$$

$$w_n \to w,\ \bar{w}_n \to w \text{ in } L^p(0,T;W^{r,1}(\Omega)),\ p > 1,\ r > 1. \tag{50}$$

The Aubin-Lions lemma ([14]) was used in (50).

If we set $B(w_i,w_i) = U_i,\ i = 0,1,...,n$ we obtain also the existence of $U \in W^{1,\infty}(0,T;V)$ such that

$$U_n \rightharpoonup^* U \quad \text{in} \quad W^{1,\infty}(0,T;V), \tag{51}$$

$$U_n(t) \rightharpoonup U(t),\ \bar{U}_n(t) \rightharpoonup U(t) \text{ in } V \ \text{ for every } t \in [0,T], \tag{52}$$

$$U_n \rightharpoonup^* U,\ \bar{U}_n \rightharpoonup^* U,\ U_n' \rightharpoonup^* U' \text{ in } L^\infty(0,T;V). \tag{53}$$

Using the properties of the bilinear operator $B : V \times V \to V$ we obtain that

$$U(t) = B(w(t),w(t)). \tag{54}$$

Let us express the discrete equations (30), (31) in a differential form

$$\begin{aligned} &w_n'(t) + a\bar{w}_n(t) + bB(U_n'(t) + a\bar{U}_n(t), \bar{w}_n(t)) \\ &\qquad = q_n'(t) + \beta\bar{q}_n(t) \ \text{ for a.e. } \ t \in (0,T] \end{aligned} \tag{55}$$

$$w_n(0) + bB(B(w_n(0),w_n(0)),w_n(0)) = q_n(0), \tag{56}$$

We now verify that the limiting function $w \in W^{1,\infty}(0,T;V)$ is a solution of the initial value problem (28), (29). We have directly from the definition of $w_0 \in V$ in (30) that

$$w_n(0) = w_0 = w(0) \ \text{ for every } n = 1,2,...$$

and the initial condition (29) is fulfilled.

Let $v \in L^2(0,T;V)$ be an arbitrary test function. The regularity $q \in C^1([0,T],V)$ and the convergence (47), (49) imply

$$\int_0^T ((\ q_n'(t) + \beta\bar{q}_n(t), v(t)\ ))dt \to \int_0^T ((\ q'(t) + \beta q(t), v(t)\ ))dt, \tag{57}$$

$$\int_0^T ((\ w_n'(t) + a\bar{w}_n(t), v(t)\ ))dt \to \int_0^T ((\ w'(t) + aw(t), v(t)\ ))dt. \tag{58}$$

Applying relations (5), (6) we have

$$((B(u,w),v)) = \int_\Omega [u,v]w dx \; \forall \; u,v,w \in V \tag{59}$$

and

$$\begin{aligned} &\int_0^T \int_\Omega [(U_n'(t) + a\bar{U}_n(t), v(t)]\bar{w}_n(t) dx dt \; \rightarrow \\ &\int_0^T \int_\Omega [(U'(t) + aU(t), v(t)]w(t) dx dt, \end{aligned} \tag{60}$$

where we applied the convergence (48), (51) and the fact that

$$[u,w] \in L^1(0,T;L^1(\Omega)) \;\; \text{for all } u,w \in L^2(0,T;V).$$

Considering the convergences (57), (58), (60) we arrive at the integral identity

$$\begin{aligned} &\int_0^T ((\; w'(t) + aw(t), v(t) \;))dt \; + \\ &\int_0^T ((\; bB(B(w(t),w(t))' + aB(w(t),w(t)), w(t)), v(t) \;))dt \; = \\ &\int_0^T ((\; q'(t) + \beta q(t), v(t) \;)) \text{ for all } v \in L^2(0,T;V), \end{aligned}$$

which implies together with (56) that w is a solution of the initial value problem (28), (29). If $\alpha < \frac{a}{2}$, then there exists τ_0 such that for $\tau \in (0,\tau_0)$ the condition (39) holds. Together with the bound (40) for the function q and its derivative all the conditions for the a priori estimates obtained above are fulfilled. We have there derived the solution $w \in W^{1,\infty}(0,T;V)$ of the problem (28), (29) as the limit of the sequences of segment line and step size functions defined in (47)-(50).

It remains to verify the uniqueness. We derive it even in the case that the bounds (40) is fulfilled only for the initial point $t = 0$:

$$\|q(0)\| < \frac{1}{\sqrt{b}\|B\|}. \tag{61}$$

Let $w_i \in W^{1,\infty}(0,T;V), i = 1,2$ be solutions of the initial value problem (28), (29). We deduce from Theorem 2.1 and from (61) the uniqueness of a solution of the stationary equation (28). The difference $w = w_1 - w_2$ then fulfils the homogeneous initial value problem

$$w'(t) + aw(t) + bB(B(w_1, w_1)' + aB(w_1, w_1), w_1)(t) \tag{62}$$
$$-bB(B(w_2, w_2)' + aB(w_2, w_2), w_2)(t) = 0,$$
$$w(0) + bB(B(w(0), w(0)), w(0)) = 0. \tag{63}$$

After multiplying with w in the space V and integrating we obtain the identity

$$\|w(t)\|^2 + a\int_0^t \|w(s)\|^2 ds = \tag{64}$$
$$b\int_0^t ((B(B(w_2, w_2)', w_2) - (B(B(w_1, w_1)', w_1), w))ds +$$
$$ba\int_0^t ((B(B(w_2, w_2), w_2) - (B(B(w_1, w_1), w_1), w))ds.$$

Let us set $w_\xi = w_2 + \xi w,\ \xi \in R$. We can then express the functions in the integrals on the right-hand side of (64) as following integrals

$$((B(B(w_2, w_2)', w_2) - (B(B(w_1, w_1)', w_1), w)) =$$
$$-\int_0^1 [((\ B(w_\xi, w_\xi)', B(w, w)\ )) + ((\ B(w, w_\xi)', B(w_\xi, w)\ ))]d\xi,$$
$$((B(B(w_2, w_2), w_2) - (B(B(w_1, w_1), w_1), w)) =$$
$$-\int_0^1 [((\ B(w_\xi, w_\xi), B(w, w)\ )) + 2\|B(w_\xi, w)\|^2]d\xi.$$

Using the fact that functions $w_i,\ i = 1, 2$ belong to the space $W^{1,\infty}(0, T; V)$ and the same holds for w_ξ we obtain from (64) the estimate

$$\|w(t)\|^2 \leq C_4 \int_0^t \|w(s)\|^2 ds$$

with the constant C_4 depending only on the $a,\ b,\ \|B\|$ and the right-hand side $q' + \beta q$. The Gronwall lemma implies $w(t) = 0,\ t \in [0, T]$ and the uniqueness of a solution follows. $\square$

After coming back to the original problem for a couple $\{w, \Phi\}$ of the deflection and the Airy stress function we obtain

Theorem 3.5 *Let $f \in C^1([0, T]; V^*)$ satisfy the inequality*

$$e^{-2\alpha t}\|f(0)\|_*^2 + \frac{1}{a}\int_0^t e^{-2\alpha(t-s)}\|f'(s) + \beta f(s)\|_*^2 ds < \frac{D_0^2}{b\|B\|^2},\ t \in [0, T] \tag{65}$$

with an arbitrary $\alpha \in (0, \frac{a}{2})$.

Then there exists a couple $\{w, \Phi\} \in W^{1,\infty}(0,T;V) \times L^{\infty}(0,T;V)$ *being a unique weak solution of the problem (20)-(24).*

Remark 3.6 *Condition (65) can be interpreted as a bound on the data. The exponential character of conditions (40) and (65) implies that the bounds for the right hand sides* q, q' *or* f, f' *do not depend on the length* T *of the time interval. We remember the relations*

$$2\alpha < a = \frac{\beta D_0}{D_0 + D_1} < \beta.$$

In the case of constant functions f *and hence also* q *we have then the following sufficient bounds for the existence and uniqueness of a solution :*

$$\|q\| < \frac{a}{\sqrt{b}\beta\|B\|} \tag{66}$$

and

$$\|f\|_* < \frac{D_0}{\sqrt{b}\|B\|}. \tag{67}$$

Remark 3.7 *Applying the Rothe's method to a weak formulation of the problem (20)-(24) means that we obtain the stationary von Kármán system at each time level. A solution is a minimizer of the functional defined in (34). We can use some of the gradient algorithms ([13]) combined with cubic finite elements in order to solve the corresponding minimum problem.*

Another possibility is to use the mixed formulation of the stationary problem due to Miyoshi [15] or [11], [17]. A weak formulation of the problem is converted into the problem involving 8 unknown functions with at most 2-nd order derivatives and using linear finite elements.

4 The Behaviour of a Solution for $t \to \infty$

Let us assume that

$$q(t) \to q_\infty,\ q'(t) \to 0 \ \text{ in } V \ \text{ for } t \to \infty.$$

It can be easily verified that in the linear case

$$w'(t) + aw(t) = q'(t) + \beta q(t),\ w(0) = q(0)$$

the relation

$$\lim_{t\to\infty} w(t) = \frac{\beta}{a} q_\infty = w_\infty$$

holds with a weak solution w_∞ of the elastic problem

$$D_0 \Delta^2 w_\infty = f_\infty \text{ in } \Omega, \quad w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma.$$

This behaviour holds also in the nonlinear case if the right-hand side of the equation (28) fulfils the bound (40) and its limit value q_∞ satisfies an estimate stronger than in (66).

Theorem 4.1 *Let $q \in C^1([0,\infty); V)$ fulfil the condition (40) with $\alpha < \frac{a}{2}$ and*

$$\lim_{t\to\infty} q(t) = q_\infty, \ \lim_{t\to\infty} q'(t) \to 0 \ \text{ in } V. \tag{68}$$

If

$$\|q_\infty\| < \frac{a}{3\sqrt{2b}\beta\|B\|}, \tag{69}$$

then

$$\lim_{t\to\infty} w(t) = w_\infty, \tag{70}$$

where $w : [0,\infty) \to V$ is a solution of the initial value problem (28), (29) and $w_\infty \in V$ fulfils the stationary equation

$$a w_\infty + abB(B(w_\infty, w_\infty), w_\infty) = \beta q_\infty. \tag{71}$$

Proof. The assumption (69) implies that the right-hand side q_∞ of the stationary von Kármán equation (71) fulfils the estimate

$$\|q_\infty\| < \frac{a}{\sqrt{b}\beta\|B\|} \tag{72}$$

Comparing with Theorem 2.1 we can see that there exists a unique solution $w_\infty \in V$ of the equation (71). Let us set

$$u(t) = w(t) - w_\infty, \ r(t) = q'(t) + \beta q(t) - \beta q_\infty, \ t \geq 0. \tag{73}$$

The function $u \in W^{1,\infty}(0,T;V)$ is for every $T > 0$ a solution of the initial value problem

$$\begin{gathered} u'(t) + au(t) + bB(B(w,w)', w)(t) + \\ ab[B(w,w), w)(t) - B(B(w_\infty, w_\infty), w_\infty)] = r(t), \end{gathered} \tag{74}$$

$$u(0) = u_0 \in V. \tag{75}$$

Let us set

$$w_\xi = (1-\xi)w_\infty + \xi w = w + (\xi - 1)u, \ \xi \in R.$$

We shall use in our further considerations the following relations:

$$\begin{gathered}((B(B(w,w),w) - B(B(w_\infty,w_\infty),w_\infty),u)) = \\ \textstyle\int_0^1 [\ ((\ B(w_\xi,w_\xi),B(u,u)\ )) + 2\|B(w_\xi,u)\|^2\ ]d\xi = \\ \tfrac{1}{3}((\ B(w,w)+B(w,w_\infty)+B(w_\infty,w_\infty),B(u,u)\ )) + \\ 2\|B(w,u)\|^2 - 2((\ B(w,u),B(u,u)\ )) + \tfrac{2}{3}\|B(w,w)\|^2.\end{gathered} \tag{76}$$

After multiplying the equation (74) we obtain considering (76) the relation

$$\begin{gathered}((u',u)) + a\|u\|^2 + 2b((\ B(w',w),B(w,u)\ )) \\ +2ab[\|B(w,u)\|^2 - ((\ B(w,u),B(u,u)\ )) + \tfrac{1}{3}\|B(u,u)\|^2] \\ +\tfrac{1}{3}ab((\ B(w,w)+B(w,w_\infty)+B(w_\infty,w_\infty),B(u,u)\ )) \\ = ((\ r(t),u(t)\ )).\end{gathered} \tag{77}$$

Further, we mention the relation

$$2((\ B(w',w),B(w,u)\ )) = \frac{d}{dt}\|B(w,u)\|^2 - 2(\ B(w',u),B(w,u)\ )). \tag{78}$$

We arrive then from (77) at the inequalities

$$\begin{gathered}\tfrac{d}{dt}[\|u\|^2 + 2b\|B(w,u)\|^2] + \tfrac{a}{2}[\|u\|^2 + 2b\|B(w,u)\|^2] \\ +\tfrac{2}{3}ab((\ B(w,w)+B(w,w_\infty)+B(w_\infty,w_\infty),B(u,u)\ )) \le \\ 4b((\ B(w',u),B(w,u)\ )) + 2((\ r(t),u(t)\ ))\end{gathered}$$

and

$$\begin{gathered}\tfrac{d}{dt}[\|u\|^2 + 2b\|B(w,u)\|^2] + \tfrac{a}{2}[\|u\|^2 + 2b\|B(w,u)\|^2] \le \\ b\|B\|^2[4\|w'\|\|w\| + \tfrac{2}{3}a(\|w\|^2 + \|w\|\|w_\infty\| + \|w_\infty\|^2)]\|u\|^2 \\ +2((\ r(t),u(t)\ )),\ t>0.\end{gathered} \tag{79}$$

We need the estimates of $w,\ w_\infty$ and w' in order to derive the conditions for the limit behaviour of u.

We have from (71) the estimate

$$\|w_\infty\| \le \frac{\beta}{a}\|q_\infty\|. \tag{80}$$

Directly from the initial value problem (28), (29) we have the relations

$$\begin{gathered}\tfrac{d}{dt}(\|w\|^2 + b\|B(w,w)\|^2)(t) + 2a\|w(t)\|^2 + b\|B(w,w)(t)\|^2 \\ = 2((\ q'(t) + \beta q(t), w(t)\ )),\ \ t > 0,\end{gathered}$$

$$\begin{gathered}\|w(t)\|^2 + b\|B(w,w)(t)\|^2 \le \\ e^{-at}\|q(0)\|^2 + \tfrac{1}{a}e^{-at}\textstyle\int_0^t e^{as}\|q'(s) + \beta q(s)\|^2 ds, \quad \text{for each } t > 0,\end{gathered}$$

$$\|w'(t)\|^2 \le 2a^2(\|w\|^2 + b\|B(w,w)\|^2)(t) + 2\|q'(t) + \beta q(t)\|^2 \ \ \text{for a.e. } t > 0.$$

Assuming the limits of the right-hand side in (68) we obtain for an arbitrary $\varepsilon > 0$ the existence of $T > 0$ such that there hold the estimates

$$\|w(t)\| \le \frac{\beta}{a}\|q_\infty\| \ + \varepsilon \quad \text{for each} \quad t > T, \tag{81}$$

$$\|w'(t)\| \le 2\beta\|q_\infty\| \ + \varepsilon \quad \text{for a.e.} \quad t > T. \tag{82}$$

Implying the estimates (80), (81), (82) in the inequality (79) we obtain that the condition (69) enables the existence of constants $c \in (0, \frac{a}{2}),\ d > 0$ and $t_0 > 0$ such that

$$\begin{gathered}\tfrac{d}{dt}[\|u\|^2 + 2b\|B(w,u)\|^2] + c[\|u\|^2 + 2b\|B(w,u)\|^2] \\ \le d\|r(t)\|^2, \quad \text{for every } t > t_0.\end{gathered} \tag{83}$$

The estimate

$$\|u\|^2 \le [\|u_0\|^2 + 2b\|B(w_0,u_0)\|^2]e^{-ct} + \int_0^t e^{-c(t-s)}\|r(s)\|^2 ds$$

then implies the limit (70) which concludes the proof. □

Acknowledgement. This work was supported by Grant 1/8263/01 of the Grant Agency of the Slovak Republic.

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