International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems
Vol. 0, No. 0 (2002) 000–000

STATES, UNIFORMITIES AND METRICS ON LATTICE EFFECT ALGEBRAS

ZDENKA RIEČANOVÁ
Department of Mathematics, Faculty of Electrical Engineering and Information Technology STU
Ilkovičova 3, SK-81219 Bratislava 1, Slovak Republic

Received
Revised

We show that every state ω on a lattice effect algebra E induces a uniform topology on E. If ω is subadditive this topology coincides with pseudometric topology induced by ω. Further, we show relations between the interval and order topology on E and topologies induced by states.

Keywords: lattice effect algebra, state, valuation, interval topology, oredr topology, topology induced by a state

1. Introduction and basic definitions

A model for an effect algebra is the standard effect algebra of positive self-adjoint operators dominated by the identity on a Hilbert space. In general an effect algebra is a partial algebra satisfying very simple axioms.

Effect algebras [6] (or, equivalent in some sense, D-posets [13], [14]) were introduced as carriers of states or probability measures in the quantum (or fuzzy) probability theory (see [10], [11], [13]). Thus elements of these structures represent quantum effects or fuzzy events which have yes-no character that may be unsharp or imprecise. Unfortunately, there are even finite effect algebras admitting no states hence also no probabilities [19]. Moreover, a state on an effect algebra need not be subadditive. It was proved in [20] that a state on a lattice effect algebra is subadditive iff it is a valuation. Further, if a faithful (i.e., non-zero at non-zero elements) valuation on an effect algebra E exists then E is modular and separable [20]. Conversely, on every complete modular atomic effect algebra there exists an (o)-continuous state [18], [21]. The aim of this paper is to bring some topological properties of lattice (or complete) effect algebras on which states, order-continuous states or valuations exist. Namely, we study properties of order and interval topologies of such effect algebras. Further we show relations of these topologies to uniform or metric topologies induced by states or valuations on them.

Definition 1.1. A structure $(E;\oplus,0,1)$ is called an *effect-algebra* if 0, 1 are two distinguished elements and $\oplus$ is a partially defined binary operation on P which satisfies the following conditions for any $a,b,c\in E$:

(i) $b\oplus a=a\oplus b$ if $a\oplus b$ is defined,
(ii) $(a\oplus b)\oplus c=a\oplus(b\oplus c)$ if one side is defined,
(iii) for every $a\in P$ there exists a unique $b\in P$ such that $a\oplus b=1$ (we put $a'=b$),
(iv) if $1\oplus a$ is defined then $a=0$.

We often denote the effect algebra $(E;\oplus,0,1)$ briefly by E. In every effect algebra E we can define the partial operation $\ominus$ and the partial order $\leq$ by putting

$$a\leq b \text{ and } b\ominus a=c \text{ iff } a\oplus c \text{ is defined and } a\oplus c=b.$$

Since $a\oplus c=a\oplus d$ implies $c=d$, the $\ominus$ and the $\leq$ are well defined. If E with the defined partial order is a lattice (a complete lattice) then $(E;\oplus,0,1)$ is called a *lattice effect algebra* (a *complete effect algebra*). It is well known that a lattice effect algebra is a common generalization of orthomodular lattices and MV-algebras (see [4] and [14]).

Lemma 1.2. Elements of an effect algebra $(E;\oplus,0,1)$ satisfy the properties:

(i) $a\oplus b$ is defined iff $a\leq b'$,
(ii) $a\leq a\oplus b$,
(iii) if $a\oplus b$ and $a\vee b$ exist then $a\wedge b$ exists and $a\oplus b=(a\wedge b)\oplus(a\vee b)$,
(iv) $a\oplus b\leq a\oplus c$ iff $b\leq c$ and $a\oplus c$ is defined,
(v) $a\ominus b=0$ iff $a=b$,
(vi) $a\leq b\leq c$ implies that $c\ominus b\leq c\ominus a$ and $b\ominus a=(c\ominus a)\ominus(c\ominus b)$.

If E is a lattice effect algebra then

(vii) $c\leq a,b \implies (a\vee b)\ominus c=(a\ominus c)\vee(b\ominus c)$ and $(a\wedge b)\ominus c=(a\ominus c)\wedge(b\ominus c)$,
(viii) $a,b\leq c \implies c\ominus(a\vee b)=(c\ominus a)\wedge(c\ominus b)$ and $c\ominus(a\wedge b)=(c\ominus a)\vee(c\ominus b)$,
(ix) $a,b\leq c' \implies (a\oplus c)\vee(b\oplus c)=(a\vee b)\oplus c$ and $(a\wedge b)\oplus c=(a\oplus c)\wedge(b\oplus c)$.

It is worth noting that if $(E;\oplus,0,1)$ is an effect algebra then $(E;\ominus,0,1)$ with the partial binary operation $\ominus$ defined above is a *D-poset* introduced by Kôpka and Chovanec [14], and vice versa.

Definition 1.3. Let $(E;\oplus,0,1)$ be an effect algebra. $Q\subseteq E$ is called a *sub-effect algebra* iff

(i) $1\in Q$,
(ii) if from elements $a,b,c\in E$ with $a\oplus b=c$ at least two are elements of Q then $a,b,c\in Q$.

For more details on D-posets and efefct algebras we refer the reader to [4].

Definition 1.4. Assume that $(E;\oplus,0,1)$ is an effect algebra. A map $m:E\to[0,1]$ is called a (finitely additive) *state* on E if $m(1)=1$ and $a\leq b' \implies m(a\oplus b)=$

$m(a) + m(b)$. We say that m is *faithful* if $m(a) = 0 \implies a = 0$.

A state m on a lattice effect algebra E is called a *valuation* if for $a, b \in E$, $a \wedge b = 0 \implies m(a \vee b) = m(a) + m(b)$.

Note that if m is a state on an effect algebra E then for $a, b \in E$ with $a \leq b$ we have $b = a \oplus (b \ominus a)$, which implies $m(b) = m(a) + m(b \ominus a)$. Thus $a \leq b \implies m(a) \leq m(b)$ and $m(b \ominus a) = m(b) - m(a)$.

If ω is a valuation on a lattice effect algebra E then evidently $\omega(a \vee b) \leq \omega(a) + \omega(b)$ for all $a, b \in E$ (we say that ω is *subadditive*). On the other hand a state on a lattice effect algebra need not be subadditive.

Theorem 1.5. [20] Assume that E is a lattice effect algebra.

(i) Every subadditive state ω on E is a valuation.

(ii) A state ω on E is a valuation iff $\omega(a \vee b) + \omega(a \wedge b) = \omega(a) + \omega(b)$ for all $a, b \in E$.

(iii) If there exists a faithful valuation ω on E then E is modular and separable.

The existence of (o)-continuous states or valuations on some families of lattice effect algebras has been proved in [18], [21], [22].

2. Uniform topologies induced by states on lattice effect algebras

If a net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of a topological space (X, τ) converges to a point $x \in X$ we will write $x_\alpha \xrightarrow{\tau} x$. Here τ denotes also the collection of all open subsets of X.

Theorem 2.1. Every state ω on a lattice effect algebra E induces a uniform topology τ_ω such that for a net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E

$$x_\alpha \xrightarrow{\tau} x \text{ iff } \omega(x_\alpha \vee y) \to \omega(x \vee y) \text{ and } \omega(x_\alpha \wedge y) \to \omega(x \wedge y) \text{ for all } y \in E\,.$$

Proof. Consider the function family $\Phi = \{\omega_{y\vee} \mid y \in E\} \cup \{\omega_{y\wedge} \mid y \in E\}$, where $\omega_{y\wedge} \colon E \to [0,1]$ and $\omega_{y\vee} \colon E \to [0,1]$ are defined by putting $\omega_{y\vee}(x) = \omega(y \vee x)$ and $\omega_{y\wedge}(x) = \omega(y \wedge x)$ for all $x \in E$. Further, consider the family of pseudometrics on E: $\Sigma_\Phi = \{\rho_{y\vee} \mid y \in E\} \cup \{\rho_{y\wedge} \mid y \in E\}$, where $\rho_{y\vee}(a,b) = |\omega_{y\vee}(a) - \omega_{y\vee}(b)|$ and $\rho_{y\wedge}(a,b) = |\omega_{y\wedge}(a) - \omega_{y\wedge}(b)|$ for all $a, b \in E$. Let us denote by $\mathcal{U}_\Phi$ the uniformity on E induced by the family of pseudometrics Σ_Φ. Further denote by τ_ω the topology compatible with the uniformity $\mathcal{U}_\Phi$. Then for every net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E

$$x_\alpha \xrightarrow{\tau_\omega} x \text{ iff } \omega(x_\alpha \vee y) \to \omega(x \vee y) \text{ and } \omega(x_\alpha \wedge y) \to \omega(x \wedge y) \text{ for all } y \in E\,.$$

□

For a deeper discuss a topology induced by a function family Φ we refer the reader to [3]. In [15] functions on D-posets with values in arbitrary uniform space without algebraic operations were treated and a Nikodym boundedness theorem and a convergence theorem were proved.

3. Pseudometric topologies on lattice effect algebras induced by subadditive states

For elements a, b of a lattice effect algebra E we set $a \triangle b = (a \vee b) \ominus (a \wedge b)$. Then the triangle inequality $a \triangle b \leq (a \triangle b) \ominus (a \wedge b)$ fails to be true in general but it does so for every valuation ω on E.

Lemma 3.1. For every valuation ω on a lattice effect algebra E and for all $a, b, c \in E$, $\omega(a \triangle b) \leq \omega(a \triangle c) + \omega(c \triangle b)$.

Proof. For any $a, b, c \in E$ we have $\omega(a\wedge c)+\omega(b\wedge c) = \omega((a\wedge c)\vee(b\wedge c))+\omega(a\wedge b\wedge c)$. Moreover, $(a\wedge c)\vee(b\wedge c) \leq c \leq (c\vee a)\wedge(c\vee b)$ which gives $\omega((a\wedge c)\vee(b\wedge c))-\omega((c\vee a)\wedge(c\vee b)) \leq 0$. Therefore $\omega(a\triangle b) = \omega(a\vee b)-\omega(a\wedge b) \leq \omega(a\vee b\vee c)-\omega(a\wedge b\wedge c) = \omega(a\vee c)+\omega(b\vee c)-\omega((a\vee c)\wedge(b\vee c))-\omega(a\wedge c)-\omega(b\wedge c)+\omega((a\wedge c)\vee(b\wedge c)) = \omega(a\triangle c)+\omega(b\triangle c)+\omega((a\wedge c)\vee(b\wedge c))-\omega((a\vee c)\wedge(b\vee c)) \leq \omega(a\triangle c)+\omega(b\triangle c)$.
□

Assume that $(\mathcal{E}; \prec)$ is a directed set and $(P; \leq)$ is a poset. A net of elements of P is denoted by $(a_\alpha)_{\alpha\in\mathcal{E}}$. If $a_\alpha \leq a_\beta$ for all $\alpha, \beta \in \mathcal{E}$ such that $\alpha \prec \beta$ then we write $a_\alpha \uparrow$. If moreover $a = \bigvee\{a_\alpha \mid \alpha \in \mathcal{E}\}$ we write $a_\alpha \uparrow a$. The meaning of $a_\alpha \downarrow$ and $a_\alpha \downarrow a$ is dual. For instance, $a \uparrow u_\alpha \leq v_\alpha \downarrow b$ means that $u_\alpha \leq v_\alpha$ for all $\alpha \in \mathcal{E}$ and $u_\alpha \uparrow a$ and $v_\alpha \downarrow b$. We will write $b \leq a_\alpha \uparrow a$ if $b \leq a_\alpha$ for all $\alpha \in \mathcal{E}$ and $a_\alpha \uparrow a$.

A net $(a_\alpha)_{\alpha\in\mathcal{E}}$ of elements of a poset $(P; \leq)$ order converges to a point $a \in P$ if there are nets $(u_\alpha)_{\alpha\in\mathcal{E}}$ and $(v_\alpha)_{\alpha\in\mathcal{E}}$ of elements of P such that

$$a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a\,.$$

We write $a_\alpha \xrightarrow{(o)} a$ *in* P (or briefly $a_\alpha \xrightarrow{(o)} a$).

Lemma 3.2. In every lattice effect algebra E

$$x_\alpha \xrightarrow{(o)} x \text{ iff } x_\alpha \triangle x \xrightarrow{(o)} 0\,; \quad x_\alpha, x \in E\,.$$

Proof (1) Evidently $x_\alpha \xrightarrow{(o)} x \implies x_\alpha \vee x \xrightarrow{(o)} x$ and $x_\alpha \wedge x \xrightarrow{(o)} x$. By the definition of (o)-convergence there are nets $(u_\alpha)_{\alpha\in E}$, $(v_\alpha)_{\alpha\in E}$ such that $x \uparrow u_\alpha \leq x_\alpha\wedge x \leq x_\alpha\vee x \leq v_\alpha \downarrow x$ which implies that $x_\alpha\triangle x = (x_\alpha\vee x)\ominus(x_\alpha\wedge x) \leq v_\alpha\ominus u_\alpha \downarrow 0$ (see [16]) and hence $x_\alpha \triangle x \xrightarrow{(o)} 0$

(2) Assume that $x_\alpha \triangle x \xrightarrow{(o)} 0$. As $(x_\alpha \vee x) \ominus x \leq x_\alpha \triangle x$ and $x \ominus (x_\alpha \wedge x) \leq x_\alpha \triangle x$ we obtain that $(x_\alpha \vee x) \ominus x \xrightarrow{(o)} 0$ and $x \ominus (x_\alpha \wedge x) \xrightarrow{(o)} 0$. It follows that $x_\alpha \vee x \xrightarrow{(o)} x$ and $x_\alpha \wedge x \xrightarrow{(o)} x$ (see [16]). Because for every α we have $x_\alpha \wedge x \leq x_\alpha \leq x_\alpha \vee x$, we conclude that $x_\alpha \xrightarrow{(o)} x$. □

Theorem 3.3. For a state ω on a lattice effect algebra E the following conditions are equivalent:

(i) ω is subadditive,

(ii) ω is a valuation,

(iii) $\rho_\omega \colon E \times E \to [0,1]$ defined by $\rho_\omega(a,b) = \omega(a \triangle b)$ is a pseudometric.

Proof For a proof of (i) $\Longleftrightarrow$ (ii) we refer to [20].

(ii) $\Longrightarrow$ (iii): By Lemma 3.1, $\rho_\omega(a,b) \leq \rho_\omega(a,c) + \rho_\omega(b,c)$ for all $a,b,c \in E$. The rest is trivial.

(iii) $\Longrightarrow$ (ii): If ρ_ω is a pseudometric then $\omega(a \triangle b) = \omega(a \vee b) - \omega(a \wedge b) = \rho_\omega(a,b) \leq \rho_\omega(a, a \wedge b) + \rho_\omega(a \wedge b, b) = \omega(a) - \omega(a \wedge b) + \omega(b) - \omega(a \wedge b)$ which gives $\omega(a \vee b) \leq \omega(a) + \omega(b)$. □

In the sequel, we will denote by τ_{ρ_ω} the pseudometric topology compatible with ρ_ω. It is easy to check that ρ_ω has the following properties:

(1) $\rho_\omega(0,a) = \rho_\omega(b, a \oplus b)$, for all $b \leq a'$

(2) $0 \leq a \leq b \implies \rho_\omega(0,b) = \rho_\omega(0,a) + \rho_\omega(a,b)$

(3) $\rho_\omega(a,b) = \rho_\omega(a \wedge b, a \vee b) = \rho_\omega(a', b')$

which gives

(4) $a_\alpha \xrightarrow{\tau_{\rho_\omega}} a$ iff $a'_\alpha \xrightarrow{\tau_{\rho_\omega}} a'$

(5) $a_\alpha \xrightarrow{\tau_{\rho_\omega}} 0$ iff $a_\alpha \oplus b \xrightarrow{\tau_{\rho_\omega}} b$ for all $a_\alpha \leq b'$.

By (5) we obtain

(6) If $b \leq b_\alpha \leq c'$ then $b_\alpha \xrightarrow{\tau_{\rho_\omega}} b$ iff $b_\alpha \ominus b \xrightarrow{\tau_{\rho_\omega}} 0$ iff $b_\alpha \oplus c \xrightarrow{\tau_{\rho_\omega}} b \oplus c$.

Theorem 3.4. For every valuation ω on a lattice effect algebra E, $\tau_{\rho_\omega} = \tau_\omega$.

Proof (1) Assume $a_\alpha \xrightarrow{\tau_\omega} a$. Then for every $x \in E$ we have $\omega(a_\alpha \vee x) \to \omega(a \vee x)$ and $\omega(a_\alpha \wedge x) \to \omega(a \wedge x)$, hence $\omega(a_\alpha \triangle a) = \omega(a_\alpha \vee a) - \omega(a_\alpha \wedge a) \to \omega(a) - \omega(a) = 0$ which is equivalent to $a_\alpha \xrightarrow{\tau_{\rho_\omega}} a$.

(2) Conversely, let $a_\alpha \xrightarrow{\tau_{\rho_\omega}} a$. Then $\omega(a_\alpha \triangle a) = (\omega(a_\alpha \vee a) - \omega(a)) + (\omega(a) - \omega(a_\alpha \wedge a)) \to 0$ which gives $\omega(a_\alpha \vee a) \to \omega(a)$ and $\omega(a_\alpha \wedge a) \to \omega(a)$. It follows that also $\omega(a_\alpha) \to \omega(a)$. Let $x \in E$ be arbitrary. Then $\omega(a \vee x) \leq \omega(a_\alpha \vee a) + \omega(x) - \omega(a \wedge x) \to \omega(a) + \omega(x) - \omega(a \wedge x) = \omega(a \vee x)$, which gives $\omega(a_\alpha \vee a \vee x) \to \omega(a \vee x)$. Further, $\omega(a_\alpha \wedge a) + \omega(x) - \omega(a \wedge x) \leq \omega((a_\alpha \wedge a) \vee x) \leq \omega(a_\alpha \vee x) \leq \omega(a_\alpha \vee a \vee x) \to \omega(a \vee x)$ and hence $\omega(a_\alpha \vee x) \to \omega(a \vee x)$ because also $\omega(a_\alpha \wedge a) + \omega(x) - \omega(a \wedge x) \to \omega(a) + \omega(x) - \omega(a \wedge x) = \omega(a \vee x)$. Moreover,

$\omega(a_\alpha \wedge x) = \omega(a_\alpha) + \omega(x) - \omega(a_\alpha \vee x) \to \omega(a) + \omega(x) - \omega(a \vee x) = \omega(a \wedge x)$. □

4. Relations between topologies τ_i, τ_o, τ_{ρ_ω} and τ_ω

Recall that the *interval topology* τ_i on a bounded poset P is a coarsest topology in which every interval $[a, b]$ is a closed set. Hence complements of finite unions of closed sets generate an open base of τ_i. The order topology τ_o on P is the finest topology in which $x_\alpha \xrightarrow{(o)} x \implies x_\alpha \xrightarrow{\tau_o} x$. Hence a set $F \subseteq P$ is a closed set in τ_o iff for every net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of F: $x_\alpha \xrightarrow{(o)} x \implies x \in F$. Further $\tau_i \subseteq \tau_o$, and if τ_i is Hausdorff then $\tau_i = \tau_o$ (see [5]). By Frink [7] the interval topology on a lattice P is compact iff P is complete.

Definition 4.1. A lattice effect algebra $(E; \oplus, 0, 1)$ is called *order continuous* ((o)-continuous for brevity) if for any net of elements of E and $x, y \in E$: $x_\alpha \uparrow x \implies x_\alpha \wedge y \uparrow x \wedge y$.

It is easily seen that in an (o)-continuous lattice effect algebra we have:
$x_\alpha \xrightarrow{(o)} x,\ y_\alpha \xrightarrow{(o)} y \implies x_\alpha \vee y_\alpha \xrightarrow{(o)} x \vee y$ and $x_\alpha \wedge y_\alpha \xrightarrow{(o)} x \wedge y$. We need only consider that $x_\alpha \uparrow x$ iff $x'_\alpha \downarrow x'$ and that in every lattice $x_\alpha \vee y_\alpha \uparrow x \vee y$. Note that (o)-continuous lattice effect algebras were also called meet continuous lattices [8].

Theorem 4.2. Let E be a lattice effect algebra and let $\omega\colon E \to [0, 1]$ be a state on E. Then

(i) ω is faithful $\implies$ τ_ω is T_2,
(ii) ω is faithful $\implies$ $\tau_i \subseteq \tau_\omega$,
(iii) ω is subadditive and faithful $\implies$ $\tau_\omega = \tau_{\rho_\omega}$ is a metric topology,
(iv) ω is subadditive and faithful $\implies$ ($\tau_{\rho_\omega} \subseteq \tau_o$ iff ω is (o)-continuous),
(v) E and ω are (o)-continuous $\implies$ $\tau_\omega \subseteq \tau_o$.

Proof (i) Assume that $x_\alpha \xrightarrow{\tau_\omega} x_1$ and $x_\alpha \xrightarrow{\tau_\omega} x_2$. If $x_1 \neq x_2$ then either $x_1 \wedge x_2 < x_1$ or $x_1 \wedge x_2 < x_2$. Let $x_1 \wedge x_2 < x_1$. Then $\omega(x_1 \wedge x_2) < \omega(x_1)$. By definition of τ_ω we have $\omega(x_\alpha \wedge x_1) \to \omega(x_1)$ and $\omega(x_\alpha \wedge x_1) \to \omega(x_2 \wedge x_1)$, a contradiction. Hence $x_1 = x_2$, which gives that τ_ω is T_2.

(ii) Let $a \le x_\alpha \le b$ and let $x_\alpha \xrightarrow{\tau_\omega} x$. Then $x_\alpha = x_\alpha \vee a = x_\alpha \wedge b$ and by definition of τ_ω we have $\omega(x_\alpha) = \omega(x_\alpha \vee a) = \omega(x_\alpha \wedge b) \to \omega(x) = \omega(x \vee a) = \omega(x \wedge b)$. As $x \wedge b \le x \le x \vee a$ and ω is faithful we conclude that $x \wedge b = x = x \vee a$ which gives $x \in [a, b]$.

(iii) As ω is a faithful valuation, we have $\tau_\omega = \tau_{\rho_\omega}$ and $\omega(x \,\triangle\, y) = 0$ iff $x \,\triangle\, y = 0$ iff $(x \vee y) \ominus (x \wedge y) = 0$ iff $x \wedge y = x \vee y$ iff $x = y$, which gives $\rho_\omega(x, y) = 0$ iff $x = y$.

(iv) Assume that ω is (o)-continuous. Then by Lemma 3.2, $x_\alpha \xrightarrow{(o)} x \implies x_\alpha \,\triangle\, x \xrightarrow{(o)} 0 \implies \omega(x_\alpha \,\triangle\, x) \to \omega(0) = 0 \implies x_\alpha \xrightarrow{\tau_{\rho_\omega}} x$. It follows by definition of τ_o that $\tau_{\rho_\omega} \subseteq \tau_o$.

Conversely, if $\tau_{\rho_\omega} \subseteq \tau_o$ then $x_\alpha \xrightarrow{(o)} x \implies x_\alpha \xrightarrow{\tau_o} x \implies x_\alpha \xrightarrow{\tau_{\rho_\omega}} x$. As ω is τ_ω-continuous and $\tau_\omega = \tau_{\rho_\omega}$, we conclude that $\omega(x_\alpha) \to \omega(x)$, hence ω is (o)-continuous.

(v) Assume that $x_\alpha \xrightarrow{(o)} x$. Then by (o)-continuity of E we have $x_\alpha \vee y \xrightarrow{(o)} x \vee y$ and $x_\alpha \wedge y \xrightarrow{(o)} x \wedge y$ for all $y \in E$. By (o)-continuity of ω we obtain $\omega(x_\alpha \vee y) \to \omega(x \vee y)$ and $\omega(x_\alpha \wedge y) \to \omega(x \wedge y)$, for all $y \in E$, which gives $x_\alpha \xrightarrow{\tau_\omega} x$. It follows $\tau_\omega \subseteq \tau_o$ by definition of τ_o. □

Definition 4.3. We say that a bounded lattice L has *separated intervals* if given any two disjoint intervals $[a, b]$, $[c, d] \subseteq L$, the lattice L can be covered by finite number of closed intervals each of which is disjoint with at least one of $[a, b]$ and $[c, d]$.

In [17] it was proved that the interval topology τ_i on a complete lattice L is Hausdorff iff L has separated intervals.

Since the partial operation $\oplus$ on an effect algebra E is associative, the existence and the meaning of $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ for elements of E is defined recurrently. $M \subseteq E$ is called an *orthogonal set* if for every finite set $\{a_1, a_2, \ldots, a_n\} \subseteq M$, $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ is defined. $Q \subseteq E$ is called a *set of mutually orthogonal elements* if any two different elements $a, b \in Q$ are orthogonal; i.e., $a \leq b'$. Evidently, every orthogonal set is a set of mutually orthogonal elements but not conversely.

Definition 4.4. An effect algebra $(E; \oplus, 0, 1)$ is called *Archimedean* if for no nonzero element $e \in E$, $ne = e \oplus e \oplus \cdots \oplus e$ (n-times) exists for every $n \in N$. E is called *separable* if it is Archimedean and every orthogonal set of elements in E is at most countable.

It was proved in [20] that if there exists a faithful state m on an effect algebra $(E; \oplus, 0, 1)$ then E is separable.

Theorem 4.5. Let E be a complete effect algebra with separated intervals.

(i) If there exists a faithful, subadditive and (o)-continuous state ω on E then

$$\tau_i = \tau_\omega = \tau_{\rho_\omega} = \tau_o$$

and (E, τ_{ρ_ω}) is a compact metric space. Moreover, E is (o)-continuous, modular and separable.

(ii) If there exists an (o)-continuous state on E and E is (o)-continuous then

$$\tau_i = \tau_\omega = \tau_o$$

and τ_ω is a compact Hausdorff topology on E.

Proof (i) By Theorem 4.2 we have $\tau_i \subseteq \tau_\omega = \tau_{\rho_\omega} \subseteq \tau_o$. As E is a complete lattice with separated intervals, the topology τ_i is T_2 [17]. It follows that $\tau_i = \tau_o$ and by the Frink theorem τ_i is compact as E is complete. For the proof that E is (o)-continuous, modular and separable, we refer the reader to [20].

(ii) By Theorem 4.2, $\tau_i \subseteq \tau_\omega \subseteq \tau_o$ and by [17] $\tau_i = \tau_o$. □

Remark 4.6. Note that the (o)-continuity of a state ω on a separable lattice effect algebra E is equivalent with σ-additivity of ω, i.e., $\omega(\bigoplus_{n=1}^{\infty} x_n) = \sum_{n=1}^{\infty} \omega(x_n)$, where $\bigoplus_{n=1}^{\infty} x_n = \bigvee_{n=1}^{\infty}(\bigoplus_{k=1}^{n} x_k)$ for every sequence $(x_n)_{n=1}^{\infty}$ in E for which $\bigvee_{n=1}^{\infty}(\bigoplus_{k=1}^{n} x_k)$ is defined.

In [22] it was proved that on every Archimedean atomic distributive effect algebra E (e.g., every Archimedean atomic MV-algebra) there exists an (o)-continuous subadditive state. By [18] on every separable complete modular atomic effect algebra E there exists an (o)-continuous faithful state. Moreover, by a generalzation of the Kaplansky theorem [18] every complete modular atomic effect algebra is an (o)-continuous lattice.

Acknowledgments

This research was supported by Grant 1/7625/20 of the Ministry of Education of the Slovak Republic.

References

1. C.C. Chang, "Algebraic analysis of many-valued logics", *Trans. Amer. Math. Soc.* **88** (1958) 467–490.
2. R. Cignoli and I.M.L. D'Ottaviano, D. Mundici, *Algebraic Foundations of Many-valued Reasoning* (Kluwer Academic Publishers, Dordrecht, 2000).
3. A. Császár, *General Topology* (Akadémiai Kiadó, Budapest, 1978).
4. A. Dvurečenskij and S. Pulmannová, *New Trends in Quantum Structures* (Kluwer Academic Publishers, Dordrecht, Boston, London and Ister Science, Bratislava, 2000).
5. M. Erné and S. Weck, "Order convergence in lattices", *Rocky Mountains Journal of Mathematics* **10** (1980) 805–818.
6. D. Foulis and M.K.Bennett, "Effect algebras and unsharp quantum logics", *Found. Phys.* **24** (1994) 1331–1352.
7. O. Frink, "Topology in lattices", *Trans. Amer. Math. Soc.* **51** (1942) 569–582.
8. G. Gierz et al., *A Compendium of Continuous Lattices* (Springer Verlag, 1980).
9. G. Grätzer, *General Laatice Theory*, second edition (Birkhäuser Verlag Basel, Boston, Berlin, 1998).
10. P. Hájek, *Mathematics of Fuzzy Logic* (Kluwer Academic Publishers, Dordrecht, 1998).
11. U. Höhle and E.P. Klement (eds.), *Non-classical Logics and their Applications to Fuzzy Subsets* (Kluwer Academic Publishers, 1995).
12. G. Kalmbach, *Orthomodular Lattices* (Academic Press London, 1983).
13. F. Kôpka, "D-posets of fuzzy sets", *Tatra Mt. Math. Publ.* **1** (1992) 83–87.
14. F. Kôpka, F. Chovanec, "Boolean D-posets" *Internat. J. Theor. Phys.* **34** (1995) 1297–1302.

15. E. Pap, *Null-Additive Set Functions* (Kluwer Academic Publishers, Dordrecht-Boston-London and Ister Science, Bratislava, 1995).
16. Riečanová, Z., "On order topological continuity of effect algebra operations" in *Proceeddings of the* 58^{th} *Workshop on General Algebra, Vienna University of Technology, June 3–6, 1999*, *Contributions to General Algebra* **12** (Verlag-Johanes Heyn Klagenfurt, 2000, pp. 349–354).
17. Riečanová, Z., "Lattice and quantum logics with separated intervals, atomiticity" *Internat. J. Theor. Phys.* **37** (1998) 191–197.
18. Z. Riečanová, "Continuous lattice effect algebras admitting order-continuous states, *Fuzzy Sets and Systems* to appear, preprint, www.elf.stuba.sk/˜jenca/preprint/index.htm
19. Z. Riečanová, "Proper effect algebras admitting no states" *Internat. J. Theor. Phys.* **40** (2001) 1683–1691.
20. Z. Riečanová, "Lattice effect algebras with (o)-continuous faithful valuations", *Fuzzy Sets and systems* **124** (2001) 321–327.
21. Z. Riečanová, "Smearings of states defined on sharp elements onto effect algebras", preprint, www.elf.stuba.sk/˜jenca/preprint/index.htm
22. Z. Riečanová, "Distributive atomic effect algebras", preprint, www.elf.stuba.sk/˜jenca/preprint/index.htm