ON THE GENERALIZED VON KÁRMÁN SYSTEM FOR THE VISCOELASTIC PLATE AND ITS SEMIDISCRETIZATION. II. LONG MEMORY CASE

Igor Bock *

Abstract

We shall deal with the system of quasistationary von Kármán equations describing great deflections of thin viscoelastic plates. We shall concentrate on a long memory material modelled by a quasistationary system with a linear integro-differential main part and a nonlinear integro-differential term. The existence and the uniqueness of a solution as the limit of a semidiscrete approximation is verified.

1 Introduction

We continue in the investigation of the behaviour of a viscoelastic isotropic plate involving the geometrical nonlinearity. We assume the bounded middle surface region Ω with a Lipschitz continuous boundary Γ. We have formulated in the first part of the paper [1] the integro-differential von Kármán system for the deflection $w(t,x)$ and the Airy stress function $\Phi(t,x),\ t \geq 0,\ x \in \Omega:$

$$D(0)\Delta^2 w + D' * \Delta^2 w - [\Phi, w] = f(t),$$
$$\Delta^2\Phi = -\frac{h}{2}(E(0)[w,w] + E' * [w,w]),$$

where $E \in C^1(R^+)$ is a positive decreasing relaxation function, $D(t) = \frac{h^3}{12(1-\mu^2)}E(t),\ t \geq 0$ is the material function, $h > 0$ the thickness of a plate,

*This work was supported by Grant 1/8263/01 of the Grant Agency of the Slovak Republic

$\mu \in (0, 1/2)$ the Poisson ratio, $(f * g)(t) = \int_0^t f(t-s)g(s)ds$ the convolution product and

$$[v, w] = \partial_{11}v\partial_{22}w + \partial_{22}v\partial_{11}w - 2\partial_{12}v\partial_{12}w, \quad v,\ w \in H^2(\Omega).$$

In the special case of the relaxation function containing only one decreasing exponential function i.e.

$$E(t) = E_0 + E_1 e^{-\beta t},\ E_0 > 0,\ E_1 > 1,\ \beta > 0$$

the original integro-differential system was transformed into the nonlinear pseudoparabolic system. We considered in [1] the clamped plate with Dirichlet boundary conditions for both the deflection and the Airy stress function. The initial-boundary value for the original system was expressed in the similar way as in the elastic case as the nonlinear initial value problem for the pseudoparabolic equation in the Sobolev space $H_0^2(\Omega)$ and solved by discretization with respect to the time variable.

We shall deal here with the general long memory isotropic case, considering the mixed boundary conditions for the deflection and the nonhomogeneous conditions for the Airy stress function formulated in a similar way as in [5] or [10]. The existence of a weak solution of the resulting nonlinear integro-differential system will be verified as the limit of the sequences of segment line and step in time functions. The main condition for the convergence is the bound on the right-hand side, which do not depend on the lenght of the time interval.

We consider the memory term also in the equation for the Airy stress function. The dynamic viscoelastic von Kármán systems are studied nowadays mainly in the framework of controllability problems. The authors (Horn and Lasiecka [6], Lagnese [8], Muñoz Rivera and Perla Menzala [9]) have considered the memory term only in the linear part of the system.

2 Formulation of the problem

We assume that a plate is subjected both to a perpendicular load f and the forces acting along the boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, where each Γ_i is either empty or $mes(\Gamma_i) > 0$. Further we assume that $\Gamma_1 \neq \emptyset$ or $\Gamma_2 \neq \emptyset$ and Γ_2 is not a segment of a straight line. The part Γ_3 contains only smooth parts.

We shall consider the following boundary value problem:

$$D(0)\Delta^2 w + D' * \Delta^2 w - [\Phi, w] = f(t), \tag{1}$$

$$w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_1, \tag{2}$$

$$w = 0, \quad \mathcal{M}(w) + k_2 \frac{\partial w}{\partial \nu} = m_2 \text{ on } \Gamma_2, \tag{3}$$

$$\mathcal{M}(w) + k_{31} \frac{\partial w}{\partial \nu} = m_3, \quad \mathcal{S}(w) + k_{32} w = t_3 \text{ on } \Gamma_3, \tag{4}$$

$$\Delta^2 \Phi - \frac{h}{2}(E(0)[w, w] + E' * [w, w]), \tag{5}$$

$$\Phi = \phi_0, \quad \frac{\partial \Phi}{\partial \nu} = \phi_1 \quad \text{on } \Gamma, \tag{6}$$

where

$$\begin{aligned}
&\mathcal{M}(w) = D(0)M(w) + D' * M(w),\\
&M(w) = \mu \Delta w + (1-\mu)(w_{,11}\nu_1^2 + 2w_{,12}\nu_1\nu_2 + w_{,22}\nu_2^2),\\
&\mathcal{S}(w) = w_{,1}\Phi_{,2\sigma} - w_{,2}\Phi_{,1\sigma} + D(0)S(w) + D' * S(w),\\
&S(w) = -\frac{\partial}{\partial \nu}\Delta w + (1-\mu)\frac{\partial}{\partial \sigma}[w_{,11}\nu_1\nu_2 - w_{,12}(\nu_1^2 - \nu_2^2) - w_{,22}\nu_1\nu_2].
\end{aligned}$$

We set

$$w_{,i} = \frac{\partial w}{\partial x_i}, \ w_{,ij} = \frac{\partial^2 w}{\partial x_i \partial x_j}, \ \Phi_{,i\sigma} = \frac{\partial}{\partial \sigma}\frac{\partial \Phi}{\partial x_i}.$$

$\nu = (\nu_1, \nu_2)$, $\sigma = (-\nu_2, \nu_1)$ are the unit outward normal and the unit tangential vector with respect to Ω respectively.

The functions $k_2 \geq 0,\ k_{3i} \geq 0,\ i = 1, 2$ satisfy the conditions

$$k_2 \in L^p(\Gamma_2),\ k_{31} \in L^p(\Gamma_3),\ p > 1,\ k_{32} \in L^1(\Gamma_3).$$

The boundary conditions (2)-(4) mean that the plate is clamped along the part Γ_1, supported (elastically if $k_2 > 0$) or loaded by a moment distribution if $m_2 \neq 0$ on Γ_2 and elastically clamped and supported if $k_{3i} > 0$ or subjected to a moment distribution and a transversal load if $k_{3i} = 0,\ i = 1, 2$.

Let us introduce following Hilbert spaces corresponding to the boundary conditions (2)-(4) and (6). We set

$$H_0^2(\Omega) = \{v \in H^2(\Omega) |\ v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma\}.$$

$H_0^2(\Omega)$ is the Hilbert space with the inner product $((.,.))_0$ and the norm $\|.\|_0$ defined by

$$((u,v))_0 = \int_\Omega \Delta u \Delta v dx, \quad \|u\|_0 = ((u,u))_0^{1/2}, \quad u, v \in H_0^2(\Omega).$$

Further we introduce the Hilbert space

$$V = \{v \in H^2(\Omega) |\ v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_1, \quad v = 0 \text{ on } \Gamma_2\}$$

with the inner product $((.,.))$ and the norm $\|.\|$ defined by

$$((u,v)) =$$
$$\int_\Omega [u_{,11}v_{,11} + 2(1-\mu)u_{,12}v_{,12} + u_{,22}v_{,22} + \mu(u_{,11}v_{,22} + u_{,22}v_{,11}]dx$$
$$+ \int_{\Gamma_2} k_2 \frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu} d\sigma + \int_{\Gamma_3} (k_{31}\frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu} + k_{32}uv)d\sigma, \tag{7}$$
$$\|u\| = ((u,u))^{1/2}, \quad u, v \in V. \tag{8}$$

We remark that the values $u,\ v$ on the boundary Γ are meant to be pointwise due to the compact imbedding $H^2(\Omega) \subset C(\overline{\Omega})$ and the values $\frac{\partial u}{\partial \nu},\ \frac{\partial v}{\partial \nu}$ on Γ are in the sense of traces. The norm defined in (8) is in the space V equivalent with the original norm

$$\|u\|_{H^2(\Omega)} = [\int_\Omega (u^2 + u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)dx]^{1/2}$$

of the Sobolev space V (see [10], Lemma 11.3.2 for the details).

We denote by V^* the space of all linear bounded functionals over V with the norm $\|f\|_*$ and the duality pairing $\langle f, v\rangle$ for $f \in V^*$ and $v \in V$.

Finally we impose the conditions upon the right-hand sides in the Problem (1)-(6). We assume

$$f \in W^{1,2}(0,T;V^*), \tag{9}$$
$$m_i \in W^{1,2}(0,T;L^p(\Gamma_i)), \quad i = 2,\ 3 \tag{10}$$
$$t_3 \in W^{1,2}(0,T;L^p(\Gamma_3)), \quad T > 0. \tag{11}$$

For any Banach space X we denote by $W^{1,2}(0,T;X)$ the space of functions $f \in L^2(0,T;X)$ such that $f' \in L^2(0,T;X)$, where f' is the derivative in the sense of distributions $\mathcal{D}'(0,T;X)$ of the function f. It can be verified in the same way as for real functions that $W^{1,2}(0,T;X)$ is a Banach space with a norm $\|f\|_{W^{1,2}} = \|f\|_{L^2(0,T;X)} + \|f'\|_{L^2(0,T;X)}$. (See [2] for further properties of the space $W^{1,2}(0,T;X)$).

The assumptions of smoothness with respect to t are due to the existence of a weak solution derived in the next chapter.

If X is a Banach space, then we denote by $C(0,T;X)$ the Banach space of continuous functions defined on the interval $[0,T]$ with values in X.

We suppose the functions $\phi_i : [0,T] \times \Gamma \to R,\ i = 0,1$ to be sufficiently smooth in order to enable the existence of a function $F \in C([0,T], H^2(\Omega))$ such that

$$F = \phi_0, \quad \frac{\partial F}{\partial \nu} = \phi_1 \quad \text{on } \Gamma, \tag{12}$$

$$((F(t), \phi))_0 = 0 \quad \text{for all } \phi \in H_0^2(\Omega). \tag{13}$$

The paper [5] contains the detailed assumptions imposed upon $\phi_0,\ \phi_1$ in order to fulfil (12), (13).

Let us introduce the trilinear form

$$\begin{aligned} &\mathcal{B}(u,v;w) = \int_\Omega [(u_{,12}v_{,2} - u_{,22}v_{,1})w_{,1} + (u_{,12}v_{,1} - u_{,11}v_{,2})w_{,2}]dx, \\ &u, v, w \in H^2(\Omega). \end{aligned} \tag{14}$$

The existence of the integral in (14) is assured due to the compact imbedding $H^2(\Omega) \subset W^{1,4}(\Omega)$. The form $\mathcal{B}$ fulfils the inequality

$$|\mathcal{B}(u,v;w)| \le \sqrt{2}|u|_{H^2(\Omega)}|v|_{W^{1,4}(\Omega)}|w|_{W^{1,4}(\Omega)}, \quad u,v,w \in H^2(\Omega) \tag{15}$$

with seminorms

$$|u|_{H^2(\Omega)} = [\int_\Omega (u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)dx]^{1/2},$$
$$|v|_{W^{1,4}(\Omega)} = [\int_\Omega (v_{,1}^4 + v_{,2}^4)dx]^{1/4}.$$

Using the integration by parts after applying the boundary conditions we formulate a weak solution of the problem (1)-(6) in a similar way as in [5] for the elastic case.

Definition 2.1 *A pair $\{w, \Phi\}$ is a weak solution of the boundary value problem (1)-(6) if*

1. $w \in C([0,T], V)$,

2. $\Phi \in C([0,T], H^2(\Omega)), \quad \Phi = \phi_0, \quad \frac{\partial \Phi}{\partial \nu} = \phi_1$ *on* Γ, *3. There hold the identities*

$$((D(0)w(t) + (D' * w)(t), v)) - \mathcal{B}(\Phi(t), w(t); v) =$$
$$\int_{\Gamma_2} m_2(t)\frac{\partial v}{\partial \nu} d\sigma + \int_{\Gamma_3} (m_3(t)\frac{\partial v}{\partial \nu} + t_3(t)v) d\sigma + \langle f(t), v\rangle, \tag{16}$$

for all $v \in V$,

$$((\Phi(t), \phi))_0 = -\frac{h}{2}\int_\Omega (E(0)[w,w](t) + (E' * [w,w])(t))\phi dx \tag{17}$$

for all $v \in H_0^2(\Omega)$.

After expressing the Airy stress function Φ in the form $\Phi = F + \Psi$, where a function F is defined in (12), (13) we can directly derive the following

Theorem 2.2 *A pair $\{w, \Phi\}$ is a weak solution of the boundary value problem (1)-(6) if and only if $\Phi = \Psi + F$ and a pair $\{w, \Psi\} \in C([0,T], V) \times C([0,T], H_0^2(\Omega))$ fulfils the identities*

$$((D(0)w(t) + (D' * w)(t), v)) - \mathcal{B}(\Psi(t) + F(t), w(t); v) =$$
$$\int_{\Gamma_2} m_2(t)\frac{\partial v}{\partial \nu} d\sigma + \int_{\Gamma_3} (m_3(t)\frac{\partial v}{\partial \nu} + t_3(t)v) d\sigma + \langle f(t), v\rangle, \tag{18}$$

for all $v \in V$,

$$((\Psi(t), \phi))_0 = -\frac{h}{2}\int_\Omega (E(0)[w,w](t) + (E' * [w,w])(t))\phi dx \tag{19}$$

for all $v \in H_0^2(\Omega)$.

Before transforming the system (18), (19) into one canonical Volterra type nonlinear integral equation in the Hilbert space V we derive some properties of the trilinear form $\mathcal{B}(.,.;.)$. We shall use a well known formula ([4])

$$\int_\Omega [u,v]\phi dx = \int_\Omega u[v,\phi] dx \quad \text{for all } u,\ v \in V,\ \phi \in H_0^2(\Omega).$$

Applying the integration by parts and the density of the sets $C_0^\infty(\Omega)$ and $C^\infty(\overline{\Omega})$ in $H_0^2(\Omega)$ and $H^2(\Omega)$ respectively we arrive at the formula

$$\mathcal{B}(u,w;v) = \mathcal{B}(v,w;u) = \mathcal{B}(w,v;u) \quad \text{for all } u,\ w \in H^2(\Omega),\ v \in H_0^2(\Omega). \tag{20}$$

The following symmetry property

$$\mathcal{B}(u,v;w) = \mathcal{B}(u,w;v) \quad \text{for all } u,\ v,\ w \in H^2(\Omega) \tag{21}$$

can be seen directly. Using the inequality (15) we obtain the inequalities

$$|\mathcal{B}(F,u;v)| \le c_1 \|F\|_{H^2(\Omega)} \|u\|_{W^{1,4}(\Omega)} \|v\| \tag{22}$$

for all $F \in H^2(\Omega),\ u,\ v \in V$,

$$|\mathcal{B}(\phi,u;v)| \le c_2 \|\phi\|_0 \|u\|_{W^{1,4}(\Omega)} \|v\| \tag{23}$$

for all $\phi \in H_0^2(\Omega),\ u,\ v \in V$,

We introduce the bilinear operators $B : H^2(\Omega) \times H^2(\Omega) \to V$ and $B_0 : V \times V \to H_0^2(\Omega)$ as solutions of equations

$$((B(u,w),v)) = \mathcal{B}(u,w;v) \quad \text{for all } v \in V, \tag{24}$$

$$((B_0(u,w),\phi))_0 = \int_\Omega [u,w]\phi dx \quad \text{for all } \phi \in H_0^2(\Omega). \tag{25}$$

Both equations are solved uniquelly, because the right-hand sides of both relations belong to the dual spaces V^* and $(H_0^2(\Omega))^*$ respectively.

The operators $B : H^2(\Omega) \times H^2(\Omega) \to V,\ B_0 : V \times V \to H_0^2(\Omega)$ are bounded (as bilinear operators) and fulfil the properties

$$\int_\Omega [u,v]\phi dx = ((B(u,v),\phi)) = ((B(v,u),\phi)) = ((B(u,\phi),v)) \tag{26}$$

for all $u,v \in V,\ \phi \in H_0^2(\Omega)$,

$$B_0(u,v) = B_0(v,u) \quad \text{for all } u,v \in V, \tag{27}$$

$$((B(B_0(u,v),w),\phi)) = ((B_0(u,v), B_0(w,\phi)\ ))_0 \tag{28}$$

for all $u,v,w,\phi \in V$,

$$\|B(u,v)\| \le c_3 \|u\|\ \|w\|_{W^{1,4}(\Omega)}, \tag{29}$$

$$\|B(u,v)\| \le \|B\|\ \|u\|\ \|v\| \tag{30}$$

for all $u,v \in H^2(\Omega)$,

$$\|B_0(u,v)\|_0 \le c_4 \|u\|_{W^{1,4}(\Omega)} \|w\|_{W^{1,4}(\Omega)} \tag{31}$$

$$\|B_0(u,v)\|_0 \le \|B_0\|\ \|u\|\ \|v\|, \tag{32}$$

for all $u,v \in V$.

Using the operator B_0 we express the function Ψ from the identity (19) in the form

$$\Psi(t) = -\frac{h}{2}[E(0)B_0(w(t),w(t)) + (E' * B_0(w,w))(t)],\ t \in [0,T]. \tag{33}$$

Let us define the function $q : [0,T] \to V$ by the relation

$$((q(t),v)) = \frac{1}{D(0)}[\int_{\Gamma_2} m_2(t)\frac{\partial v}{\partial \nu}d\sigma + \int_{\Gamma_3}(m_3(t)\frac{\partial v}{\partial \nu} + t_3(t)v)d\sigma + \langle f(t),v\rangle] \quad (34)$$

for all $v \in V$.

The elements $q(t) \in V$ are uniquelly defined as the Riesz representants of the right-hand side in the relation (34) which is for every $t \in [0,T]$ the linear continuous functional over V. Moreover we have the regularity

$$q \in W^{1,2}(0,T;V) \quad (35)$$

due to the assumptions (9)-(11).

After inserting the values $\Psi(t)$, $t \in [0,T]$ from (33) into (18) and using the relations (24), (34) we arrive at

The canonical Volterra integral equation

$$w(t) + g * w - \alpha B(F(t),w(t)) + \alpha B(B_0(w,w)(t) + g * B_0(w,w)(t), w(t)) = q(t) \in V, \quad (36)$$

$$g(t) = \frac{D'(t)}{D(0)} = \frac{E'(t)}{E(0)}, \quad \alpha = \frac{h}{2}\frac{E(0)}{D(0)}.$$

It can be readily seen that a function w is a solution of the canonical equation (36) if and only if a pair $\{w, B_0(w,w)\} : [0,T] \to V \times H_0^2(\Omega)$ is a solution of the identities (18), (19) and hence a pair $\{w, F + B_0(w,w)\}$ is a weak solution of the original problem (1)-(6).

3 Approximation by the Rothe Method

We shall verify the existence of a solution of the canonical integro-differential equation (36) using its discretization with respect to the time variable t.

Before formulating the discrete scheme let us set some additional growth assumptions on the kernel function g and the bounds on the function F. We assume the exponential behaviour of the continuous kernel function g:

$$0 < -g(t) \leq Ke^{-\beta t}, \; t \geq 0, \; 0 < K < \beta. \quad (37)$$

corresponding to the most of viscoelastic materials (see [3] for example). Further we assume that

$$((B(F(t),v),v)) \leq 0 \;\text{ for all } v \in V, \; t \in [0,T]. \quad (38)$$

Comparing with (14), (24) we can see that the condition

$$\int_\Omega [F_{,22}(t)(v_{,1})^2 - 2F_{,12}(t)v_{,1}v_{,2} + F_{,11}(t)(v_{,2})^2]dx \geq 0 \tag{39}$$
$$\text{for all } t \in [0,T] \text{ and } v \in V$$

is sufficient for fulfilling (38).

For a fixed integer N we set

$$\tau = \tfrac{T}{N},\ t_i = i\tau,\ w_i = w(t_i),\ i = 0, 1, ..., N;$$
$$\delta w_j = \tfrac{1}{\tau}(w_j - w_{j-1}),\ j = 1, ..., N.$$

We convert the nonlinear Volterra integral equation (36) into the finite sequence of nonlinear stationary equations in the Hilbert space V similar to the canonical von Kármán equations for the elastic plate. We shall use the Rothe method in a similar way as by Kačur [7] or Slodička [11] in the case of parabolic integro-differential equations. Substituting the values w_i instead of $w(t_i),\ i = 0, 1, ..., N$ and the integrals by finite sums we arrive at the equations

$$w_0 - \alpha B(F_0, w_0) + \alpha B(B_0(w_0, w_0), w_0) = q_0, \tag{40}$$

$$w_i - \alpha B(F_i, w_i) + \tau \sum_{j=0}^{i-1} g_{i-j} w_j +$$
$$\alpha B\left(B_0(w_i, w_i) + \tau \sum_{j=0}^{i-1} g_{i-j} B_0(w_j, w_i)\right) = q_i, \tag{41}$$
$$i = 1, ..., N. \tag{42}$$

The equations (40), (41) are the Euler equations for the functionals

$$J_0(v) = \frac{1}{2}[\|v\|^2 - \alpha((B(F_0, v), v))] + \frac{\alpha}{4}\|B_0(v, v)\|_0^2$$
$$J_i(v) = \frac{1}{2}[\|v\|^2 - \alpha((B(F_i, v), v))] + \frac{\alpha}{4}\|B_0(v, v)\|_0^2 +$$
$$\left(\left(\tau \sum_{j=0}^{i-1} g_{i-j} w_j, v\right)\right) + \frac{\alpha}{2}\left(\left(\tau \sum_{j=0}^{i-1} g_{i-j} B_0(w_j, w_j), B_0(v, v)\right)\right)_0$$
$$-((q_i, v)),\ v \in V,\ i = 1, ..., n.$$

The functionals $J_i,\ i = 0, 1, ..., N$ are weakly lower semicontinuous and coercive over V. The coerciveness

$$\lim_{\|v\| \to +\infty} J_i(v) = +\infty$$

can be seen directly. The weakly lower semicontinuity is the consequence of the inequality (31) and the compact imbedding $V \subset W^{1,4}(\Omega)$ which imply

$$v_n \rightharpoonup v \text{ in } V \Longrightarrow B_0(v_n, v_n) \to B_0(v, v) \text{ in } H_0^2(\Omega). \tag{43}$$

Then there exist elements $w_i \in V$ fulfilling the minimum condition

$$J_i(w_i) = \min_{v \in V} J_i(v),\ i = 0, 1, ..., N$$

and solving the discrete canonical equations (40), (41).

We proceed with a priori estimates. In order to achieve the best possible uniform estimates we multiply the discrete canonical equations (41) with the exponential functions with positive exponents. Let

$$\gamma > 0, \quad \beta - \gamma > K \tag{44}$$

We set $i = j$ in (41), multiply it with $\tau e^{\gamma\tau j} w_j$ in V and add for $j = 0, 1, ..., i$. After denoting

$$\omega_j = \|w_j\|^2 + \alpha \|B_0(w_j, w_j)\|_0^2 \tag{45}$$

we obtain subsequently the inequalities

$$\sum_{j=0}^{i} \tau e^{\gamma j \tau} \omega_j \leq \sum_{j=1}^{i} \tau^2 e^{\gamma j \tau} \left(\left(\sum_{k=0}^{j-1} g_{j-k} w_k, w_j \right) \right)$$
$$+ \alpha \sum_{j=1}^{i} \tau^2 e^{\gamma j \tau} \left(\left(\sum_{k=0}^{j-1} g_{j-k} B_0(w_k, w_k), B_0(w_j, w_j) \right) \right)_0$$
$$+ \sum_{j=0}^{i} \tau e^{\gamma j \tau} ((q_j, w_j)),$$
$$\sum_{j=0}^{i} \tau e^{\gamma j \tau} \omega_j \leq$$
$$\frac{\tau^3}{1-\epsilon} \sum_{j=1}^{i} e^{\gamma j \tau} \left[\| \sum_{k=0}^{j-1} g_{j-k} w_k \|^2 + \alpha \| \sum_{k=0}^{j-1} g_{j-k} B_0(w_k, w_k) \|_0^2 \right]$$
$$+ \frac{\tau}{\epsilon(1-\epsilon)} \sum_{j=0}^{i} e^{\gamma j \tau} \|q_j\|^2,\ 0 < \epsilon < 1.$$

Applying the assumption (37) and the convexity of the function $\|.\|^2$ we obtain the estimates

$$\sum_{j=0}^{i} \tau e^{\gamma j\tau}\omega_j \le \tag{46}$$

$$\frac{K^2}{1-\epsilon}\tau^3 \sum_{j=1}^{i} e^{(\gamma-2\beta)j\tau} \sum_{k=0}^{j-1} e^{\beta k\tau} \sum_{k=0}^{j-1} e^{\beta k\tau}\omega_k + \frac{\tau}{\epsilon(1-\epsilon)} \sum_{j=0}^{i} e^{\gamma j\tau}\|q_j\|^2$$

$$= \frac{K^2}{1-\epsilon}\ \tau^3 \sum_{j=1}^{i} \frac{e^{-(\beta-\gamma)j\tau} - e^{-(2\beta-\gamma)j\tau}}{e^{\beta\tau}-1} \sum_{k=0}^{j-1} e^{\beta k\tau}\omega_k$$

$$+\frac{\tau}{\epsilon(1-\epsilon)} \sum_{j=0}^{i} e^{\gamma j\tau}\|q_j\|^2.$$

We continue with the estimate of the double sum in the last inequality. We have

$$\sum_{j=1}^{i}(e^{-(\beta-\gamma)j\tau} - e^{-(2\beta-\gamma)j\tau}) \sum_{k=0}^{j-1} e^{\beta k\tau}\omega_k \le \sum_{j=1}^{i}\sum_{k=0}^{j-1} e^{\beta k\tau} e^{(\gamma-\beta)j\tau}\omega_k$$

$$= \sum_{k=0}^{i-1} e^{\beta k\tau}[\sum_{j=k+1}^{i} e^{(\gamma-\beta)j\tau}]\omega_k$$

$$= e^{-(\beta-\gamma)\tau} \sum_{k=0}^{i-1} e^{\gamma k\tau} \frac{1-e^{(\gamma-\beta)(i-k)\tau}}{1-e^{(\gamma-\beta)\tau}}$$

$$\le \sum_{k=0}^{i-1} e^{\gamma k\tau} \frac{1}{e^{(\beta-\gamma)\tau}-1}\omega_k \le \frac{1}{(\beta-\gamma)\tau} \sum_{k=0}^{i-1} e^{\gamma k\tau}\omega_k.$$

Comparing with (46) we obtain the inequality

$$\sum_{j=0}^{i} \tau e^{\gamma j\tau}\omega_j \le \tag{47}$$

$$\frac{K^2}{(1-\epsilon)\beta(\beta-\gamma)} \sum_{j=0}^{i-1} \tau e^{\gamma j\tau}\omega_j + \frac{1}{\epsilon(1-\epsilon)} \sum_{j=0}^{i} \tau e^{\gamma j\tau}\|q_j\|^2.$$

Setting

$$\epsilon = 1 - \frac{K}{\sqrt{\beta(\beta-\gamma)}}$$

we obtain

$$\frac{K^2}{(1-\epsilon)\beta(\beta-\gamma)} = 1-\epsilon.$$

The inequality (47) then implies the following summation estimate:

$$\sum_{j=0}^{i} \tau e^{\gamma j\tau}\omega_j = \sum_{j=0}^{i} \tau e^{\gamma j\tau}[\|w_j\|^2 + \alpha\|B_0(w_j,w_j)\|_0^2]$$

$$\leq C_1(\beta,\gamma,K)\sum_{j=0}^{i} \tau e^{\gamma j\tau}\|q_j\|^2 \quad \text{for every } \tau > 0, \tag{48}$$

$$C_1(\beta,\gamma,K) = \frac{[\beta(\beta-\gamma)]^{3/2}}{K[\sqrt{\beta(\beta-\gamma)}-K]^2}.$$

Remark 3.1 *It was possible to use simpler approach in the obtaining the summation estimate. We could apply the discrete Gronwall lemma, but with significantly larger constant C_1, containing the length T of the time interval in (48).*

We continue with uniform a priori estimates. The equations (41), (45) imply the identity

$$\omega_i = \alpha((B(F_i,w_i),w_i)) - \left(\left(\tau\sum_{j=0}^{i-1} g_{i-j}w_j, w_i\right)\right)$$

$$-\alpha\left(\left(\tau\sum_{j=0}^{i-1} g_{i-j}B_0(w_j,w_j), B_0(w_i,w_i)\right)\right)_0 + ((q_i,w_i))$$

and employing the property (38) the inequality

$$\omega_i \leq 2\tau^2\left(\left\|\sum_{j=0}^{i-1} g_{i-j}w_j\right\|^2 + \alpha\left\|\sum_{j=0}^{i-1} g_{i-j}B_0(w_j,w_j)\right\|_0^2\right) + 2\|q_i\|^2. \tag{49}$$

Again using the convexity of $\|.\|^2$ and the properties of exponential functions we arrive at the inequality

$$e^{\gamma i\tau}\left\|\sum_{j=0}^{i-1} g_{i-j}w_j\right\|^2 = \left\|\sum_{j=0}^{i-1} g_{i-j}e^{\gamma(i-j)\tau/2}(e^{\gamma j\tau/2}w_j)\right\|^2$$

$$\leq \sum_{j=0}^{i-1}(-g_{i-j}e^{\gamma(i-j)\tau/2})\ \sum_{j=0}^{i-1}(-g_{i-j}e^{\gamma(i-j)\tau/2})e^{\gamma j\tau}\|w_j\|^2 \tag{50}$$

$$\leq K^2 \sum_{j=0}^{i-1} e^{-(\beta-\gamma/2)(i-j)\tau} \ \sum_{j=0}^{i-1} e^{-(\beta-\gamma/2)(i-j)\tau} e^{\gamma j\tau} \|w_j\|^2$$

$$\leq K^2 \frac{1-e^{-(\beta-\gamma/2)i\tau}}{e^{(\beta-\gamma/2)\tau}-1} \ \sum_{j=0}^{i-1} e^{\gamma j\tau}\|w_j\|^2 \leq \frac{K^2}{(\beta-\gamma/2)\tau} \sum_{j=0}^{i-1} e^{\gamma j\tau}\|w_j\|^2.$$

In the same way we obtain

$$e^{\gamma i\tau} \left\| \sum_{j=0}^{i-1} g_{i-j} B_0(w_j, w_j) \right\|_0^2 \leq \frac{K^2}{(\beta-\gamma/2)\tau} \sum_{j=0}^{i-1} e^{\gamma j\tau} \|B_0(w_j, w_j)\|_0^2. \tag{51}$$

Combining (45), (49), (50), (51) we obtain the inequality

$$\omega_i \ \leq \ \frac{4K^2}{2\beta-\gamma}\tau \sum_{j=0}^{i-1} \omega_j e^{-\gamma(i-j)\tau} \ + 2\|q_i\|^2$$

and applying the estimate (48) we obtain

$$\begin{aligned} &\|w_i\|^2 + \alpha\|B_0(w_i, w_i)\|_0^2 \leq \\ &C_2(\beta,\gamma,K) \sum_{j=0}^{i} \tau e^{-\gamma(i-j)\tau}\|q_j\|^2 + 2\|q_i\|^2, \ i = 1, ..., n, \end{aligned} \tag{52}$$

where

$$C_2(\beta,\gamma,K) = \frac{4K^2}{2\beta-\gamma} C_1(\beta,\gamma,K).$$

Using the boundedness of the function q in the space $C([0,T],V)$ we arrive at the estimate

$$\|w_i\|^2 + \alpha\|B_0(w_i, w_i)\|_0^2 \leq C_3(\beta,\gamma,K) \max_{t\in[0,T]} \|q(t)\|^2, \ i = 1, 2, ..., n \tag{53}$$

with

$$C_3(\beta,\gamma,K) = \frac{1}{\gamma} C_2(\beta,\gamma,K) + 2.$$

In order to achieve the convergence of the scheme we need the summation estimates of the differences δw_i.

After setting $i = j$, $i = j-1$ in (41) and substracting we have the identities

$$\delta w_1 + g_1 w_0 + \alpha\delta B(B_0(w_1, w_1) - F_1, w_1) + \alpha B(g_1 B_0(w_0, w_0), w_1) = \delta q_1,$$

$$\delta w_j + \alpha\delta B(B_0(w_j,w_j) - F_j, w_j) + g_j w_0 + \tau\sum_{k=1}^{j-1} g_{j-k}\delta w_k +$$

$$\tau\alpha B(g_j B_0(w_0,w_0) + \sum_{k=1}^{j-1} g_{j-k}\delta B_0(w_k,w_k), w_j) = \delta q_j, \quad j = 2,...,i.$$

After multiplying the last identities in the space V with $\tau\delta w_j,\ j = 1,...,i$ and adding we arrive at

$$\begin{aligned}
&\tau\sum_{j=1}^{i}\|\delta w_j\|^2 - \alpha\sum_{j=1}^{i}((B(F_j,w_j) - B(F_{j-1},w_{j-1})\ ,\ \delta w_j))\\
&+\alpha\sum_{j=1}^{i}((B(B_0(w_j,w_j),w_j) - B(B_0(w_{j-1},w_{j-1}),w_{j-1})\ ,\ \delta w_j))\\
&+\tau\sum_{j=1}^{i}((\ g_j w_0 + \alpha g_j B(B_0(w_0,w_0),w_j), \delta w_j)\ )) \qquad (54)\\
&+\tau^2\sum_{j=2}^{i}((\sum_{k=1}^{j-1} g_{j-k}\delta w_k + \alpha B(g_{j-k}\delta B_0(w_k,w_k),w_j)\ ,\ \delta w_j))\\
&= \tau\sum_{j=1}^{i}((\delta q_j, \delta w_j)).
\end{aligned}$$

Let us set

$$w_\xi = w_{j-1} + \xi(w_j - w_{j-1}),\ \xi \in R$$

for a fixed $j \in 1,...,i$. We have then the relation

$$\begin{aligned}
&((B(B_0(w_j,w_j),w_j) - B(B_0(w_{j-1},w_{j-1}),w_{j-1})\ ,\ \delta w_j)) =\\
&\tau\int_0^1 [2\|B_0(\delta w_j,w_\xi)\|_0^2 + ((\ B_0(w_\xi,w_\xi), B_0(\delta w_j,\delta w_j)\ ))_0]d\xi. \qquad (55)
\end{aligned}$$

Using the assumption (38) and the relation (55) we obtain from (54) for an arbitrary $\epsilon \in (0, \frac{1}{6})$ the inequality

$$\begin{aligned}
&(1-5\epsilon)\tau\sum_{j=1}^{i}\|\delta w_j\|^2 \ \le \alpha\|B_0\|^2 \max_{j\in\{0,\dots,i\}}\|w_j\|^2\ \tau\sum_{j=1}^{i}\|\delta w_j\|^2\\
&+\frac{1}{4\epsilon}\alpha\|B\| \max_{j\in\{0,\dots,i\}}\|w_j\|^2\tau\sum_{j=1}^{i}\|\delta F_j\|^2 \qquad (56)
\end{aligned}$$

$$+\frac{1}{4\epsilon}\tau\sum_{j=1}^{i}[g_j^2\|w_0+\alpha B(B(w_0,w_0),w_j)\|^2+\|\delta q_j\|^2]$$

$$+\frac{1}{4\epsilon}(1+2\|B\|\|B_0\|\max_{j\in\{0,\dots,i\}}\|w_j\|^2)\tau^3\sum_{j=2}^{i}\|\sum_{k=1}^{j-1}g_{j-k}\delta w_k\|^2.$$

Let us assume that

$$\alpha\|B_0\|^2\|w_j\|^2\le 1-6\epsilon \text{ for } \ j=1,...,i. \tag{57}$$

Comparing with the a priori estimates (53), we can see that the condition

$$\max_{t\in[0,T]}\|q(t)\|^2 \ < \ \Big[\alpha\|B_0\|^2C_3(\beta,\gamma,K)\Big]^{-1} \tag{58}$$

is sufficient for fulfilling the estimate (57) with some constant $\epsilon\in(0,\frac{1}{6})$.

After evaluating of the constants $C_1(\beta,\gamma,K)$, $C_2(\beta,\gamma,K)$, $C_3(\beta,\gamma,K)$ in the previous a priori estimates we obtain the condition (58) in a form

$$\max_{t\in[0,T]}\|q(t)\|^2< \ C_4(\beta,\gamma,K), \tag{59}$$

$$C_4(\beta,\gamma,K)=\frac{1}{2\alpha\|B_0\|^2}\left[1+\frac{2K(\beta^2-\beta\gamma)^{3/2}}{(2\beta\gamma-\gamma^2)(\sqrt{\beta^2-\beta\gamma}-K)^2}\right]^{-1}.$$

Applying the assumption (57), the a priori estimate (53), the properties of the function g and the differentiability assumption $q\in W^{1,2}(0,T;V)$ we obtain the inequality

$$\tau\sum_{j=1}^{i}\|\delta w_j\|^2 \ \le C_5+C_6\tau^2\sum_{j=1}^{i}\sum_{k=1}^{j-1}\|\delta w_k\|^2, \ i=1,...N. \tag{60}$$

The discrete Gronwall lemma then implies the a priori estimate

$$\tau\sum_{j=1}^{i}\|\delta w_j\|^2 \ \le C_7, \ i=1,...,N, \ \tau=\frac{T}{N} \tag{61}$$

with a constant $C_7\equiv C_7(T)$.

In order to perform the convergence analysis we introduce the increasing sequence $\{N_n\}$, $\lim_{n\to\infty}N_n=\infty$. We set

$$\tau_n=\frac{T}{N_n}, \ t_i^n=i\tau_n, \ u_i^n=u(t_i^n), \ i=0,1,...,N_n, \ u:[0,T]\to X,$$

X − any normed space ,

$$w_0^n=w_0, \ w_i^n=w_i, \ \delta w_j^n=\frac{1}{\tau}(w_j^n-w_{j-1}^n) \ i=1,...,N_n,$$

where $w_i \in V$ is a solution of the equation (41) with $\tau = \tau_n,\ F_i = F_i^n,\ g_{i-j} = g_{i-j}^n,\ q_i = q_i^n$.

Let us further define the following segmentline and step functions determined by values $w_i^n,\ \delta w_i^n$:

$$\begin{aligned}
&w_n : [0,T] \to V,\ w_n(t) = w_{i-1}^n + (t - t_i^n)\delta w_i^n,\ t_{i-1}^n \le t \le t_i^n,\\
&\bar{w}_n : [0,T] \to V,\ \bar{w}_n(0) = w_0,\ \bar{w}_n(t) = w_i^n,\ t_{i-1}^n < t \le t_i^n,\\
&\tilde{w}_n : [0,T] \to V,\ \tilde{w}_n(0) = 0,\ \tilde{w}_n(t) = w_{i-1}^n,\ t_{i-1}^n < t \le t_i^n,\\
&i = 1, ..., N(n).
\end{aligned}$$

The a priori estimates (53), (61) imply that the sequence of functions $\{w_n\}$ defined by their discrete values is bounded in the space $W^{1,2}(0,T;V)$:

$$\|w_n\|_{W^{1,2}(0,T;V)} \le C_8,\ n \in N. \tag{62}$$

Then there exists its subsequence (again denoted by $\{w_n\}$) and a function $w \in W^{1,2}(0,T;V)$ such that

$$w_n \rightharpoonup w \quad \text{in} \quad W^{1,2}(0,T;V), \tag{63}$$

$$w_n(t) \rightharpoonup w(t),\ \bar{w}_n(t) \rightharpoonup w(t) \text{ in } V \text{ for every } t \in [0,T], \tag{64}$$

$$w_n \rightharpoonup^* w,\ \bar{w}_n \rightharpoonup^* w \text{ in } L^\infty(0,T;V), \tag{65}$$

$$w_n \to w,\ \bar{w}_n \to w \text{ in } L^p(0,T;W^{1,r}(\Omega)),\ p > 1,\ r > 1. \tag{66}$$

Let us introduce the discrete values of the Airy stress function Ψ by

$$\Psi_i^n = -\alpha D(0)[B_0(w_i^n, w_i^n) + \tau \sum_{j=0}^{i-1} g_{i-j}^n B_0(w_j^n, w_j^n)], \tag{67}$$

$$i = 1, ..., N(n),\ n = 1, 2, ...$$

The corresponding sequence $\bar{\Psi}_n$ of step functions is due to the inequality (51) and the estimates (48), (52). bounded in the space $L^\infty(0,T;H_0^2(\Omega))$:

$$\|\bar{\Psi}_n\|_{L^\infty(0,T;H_0^2(\Omega))} \le C_9,\ n = 1, 2, ... \tag{68}$$

Then there exists a subsequence (again denoted by $\bar{\Psi}_n$) and a function $\Psi \in L^\infty(0,T;H_0^2(\Omega))$ such that

$$\bar{\Psi}_n \rightharpoonup^* \Psi \text{ in } L^\infty(0,T;H_0^2(\Omega)). \tag{69}$$

We shall verify that a function Ψ is determined by the expression

$$\Psi = -\alpha D(0)[B_0(w,w) + g * B_0(w,w)]. \tag{70}$$

Let us set

$$B_0(w,w) = U,\ B_0(w_n,w_n) = U_n,\ n = 1,2,...$$

We can express the functions $\bar{\Psi}_n$ in a following way:

$$\bar{\Psi}_n(t) = -\alpha D(0)\left[\bar{U}_n(t) + \int_0^t g(t-s)\tilde{U}_n(s)ds\right] + \tag{71}$$

$$\alpha D(0)\left[\int_t^{t_i^n} g(t-s)\tilde{U}_n(s)ds + \int_0^{t_i^n}(g(t-s) - g(t_i^n - s))\tilde{U}_n(s)ds\right],$$

$$t_{i-1}^n < t \le t_i^n,\ i = 1,...,N(n).$$

Applying the property (31), the convergence (66) and the boundedness of $\{\bar{w}_n\}$ $\{\tilde{w}_n\}$ in $L^\infty(0,T;V)$ and hence also in $L^\infty(0,T;W^{1,4}(\Omega))$ we obtain the convergence

$$\bar{U}_n \to U \text{ in } L^p(0,T;H_0^2(\Omega)), \tag{72}$$

$$\tilde{U}_n \to U \text{ in } L^p(0,T;H_0^2(\Omega))\ \forall p > 1. \tag{73}$$

The operator $G : L^p(0,T;H_0^2(\Omega)) \to L^p(0,T;H_0^2(\Omega))$ defined by

$$(Gu)(t) = \int_0^t g(t-s)u(s)ds,\ u \in L^p(0,T;H_0^2(\Omega));$$

is linear and continuous and the convergence

$$G\tilde{U}_n \to GU \text{ in } L^p(0,T;H_0^2(\Omega)) \tag{74}$$

follows.

The function defined by the sum of the second and the third integral in (71) converges strongly to 0 in $L^p(0,T;H_0^2(\Omega))$ as a consequence of previous a priori estimates and properties of the function g. Then we obtain using (72), (74) the relations (69), (70). Moreover we have the strong convergence

$$\bar{\Psi}_n \to \Psi \text{ in } L^p(0,T;H_0^2(\Omega)),\ p > 1. \tag{75}$$

The equations (41) for $i = 1,...,N(n)$ can be expressed in a form

$$\bar{w}_n(t) - \frac{1}{D(0)}B(\bar{F}_n + \bar{\Psi}_n, \bar{w}_n)(t) + G\tilde{w}_n(t) + \tag{76}$$

$$\int_t^{t_i^n} g(t-s)\tilde{w}_n(s)ds + \int_0^{t_i^n}(g(t-s) - g(t_i^n - s))\tilde{w}_n(s)ds = q_n(t),$$

$$t_{i-1}^n < t \le t_i^n,\ i = 1,...,N(n).$$

Applying the convergence (66), (75), the continuity of the functions $F : [0,T] \to H^2(\Omega)$, $q : [0,T] \to V$ and the relation (70) we obtain in the same way as above that the function w fulfils the canonical equation (36). If the regularity assumptions used above are fulfilled then the pair $\{w, \Phi\}$, $\Phi = \Psi + F$ is a weak solution of the boundary value problem (1)-(6). We involve this result in the following theorem.

Theorem 3.2 *Let $f \in W^{1,2}(0,T;V^*)$, $m_i \in W^{1,2}(0,T;L^p(\Gamma_i))$, $i = 2,3$, $t_3 \in W^{1,2}(0,T;L^p(\Gamma_3))$ be such that the function $q : [0,T] \to V$ defined by (35) fulfils the estimate (59). Let the function $F : [0,T] \to H^2(\Omega)$ defined in (12), (13) fulfil the condition (39) and the positive relaxation function $E \in C^1(R^+)$ fulfil the growth condition*

$$0 < -E'(t) \le KE(0)e^{-\beta t},\ t \ge 0,\ 0 < K < \beta - \gamma,\ \gamma > 0. \qquad (77)$$

Let $\{\bar{w}_n, \bar{\Psi}_n\}$ be a sequence of step functions achieved from the discrete values defined by the system (40), (41) and the relation (67). Then there exists its subsequence (again denoted by $\{\bar{w}_n, \bar{\Psi}_n\}$ which fulfils the convergence (63)-(66), (69), (75) and a couple

$$\{w, \Phi\} = \{w, \Psi + F\} \in W^{1,2}(0,T;V) \times C([0,T];H^2(\Omega))$$

is a weak solution of the von Kármán system (1)- (6).

If moreover

$$\|q(t)\|^2 < \frac{(\beta - K)^2}{\alpha \|B_0\|^2 \beta(\beta + K)} \text{ for all } t \in [0,T], \qquad (78)$$

then there exists a unique weak solution of (1)-(6) and the whole sequence $\{\bar{w}_n, \bar{\Psi}\}$ is convergent.

Proof. The growth condition (77) implies the condition (37). Our previous results then verify the existence of a solution $\{w, \Phi\} = \{w, \Psi + F\} \in W^{1,2}(0,T;V) \times C([0,T];H^2(\Omega))$, where the couple $\{w, \Psi\}$ is a limit of a subsequence from the sequence of step functions mentioned in the assertion of the theorem.

Let the assumption (78) hold. We shall verify the uniqueness of a solution which implies that the convergence (63)-(66), (69), (75) holds for the whole sequence $\{w_n, \bar{\Psi}_n\}$.

Let $\{u_i, \Phi_i\},\ i = 1, 2$ are two weak solution of the system (1)- (6). Every function $u_i,\ i = 1,\ 2$ the fulfils the canonical equation

$$
\begin{aligned}
& u_i(t) - \alpha B(F(t), u_i(t)) + (g * u_i)(t) \\
& + \alpha B[B_0(u_i, u_i)(t) + (g * B_0(u_i, u_i))(t), u_i(t)] = \ q(t),\ t \in [0, T].
\end{aligned} \tag{79}
$$

The difference $u = u_2 - u_1$ then fulfils the equation

$$
\begin{aligned}
& u(t) - \alpha B(F(t), u(t)) + (g * u)(t) \\
& + \alpha B\left(B_0(u_2, u_2)(t) + (g * B_0(u_2, u_2))(t), u_2(t)\right) \\
& - \alpha B\left(B_0(u_1, u_1)(t) + (g * B_0(u_1, u_1))(t), u_1(t)\right) \ = \ 0.
\end{aligned}
$$

Let $u_\xi = u_1 + \xi(u_2 - u_1),\ \xi \in R$. There hold the relations

$$
\begin{aligned}
& ((\ B(B_0(u_2, u_2), u_2)(t) - B(B_0(u_1, u_1), u_1)(t), u(t)\ )) = \\
& \int_0^1 [2\|B_0(u(t), u_\xi(t))\|_0^2 + ((\ B_0(u_\xi, u_\xi)(t), B_0(u(t), u(t))\ ))_0] d\xi,
\end{aligned} \tag{80}
$$

$$
\begin{aligned}
& ((\ B(g * B_0(u_2, u_2))(t), u_2(t)) - B(g * B_0(u_1, u_1))(t), u_1)(t)), u(t)\ )) = \\
& 2 \int_0^1 ((\ g * B_0(u, u_\xi)(t)), B_0(u, u_\xi)(t)\ ))_0 d\xi \\
& + \int_0^1 ((\ (g * B_0(u_\xi, u_\xi))(t), B_0(u, u)(t)\ ))_0 d\xi.
\end{aligned} \tag{81}
$$

Using the assumption (39) and the relations (80), (81) we obtain the inequality

$$
\begin{aligned}
& \|u\|^2 + ((\ \int_0^t g(t-s) u(s) ds, u(t)\ )) \\
& + \alpha \int_0^1 [2\|B_0(u(t), u_\xi(t))\|_0^2 + ((\ B_0(u_\xi, u_\xi)(t), B_0(u(t), u(t))\ ))_0] d\xi \\
& + \alpha \int_0^1 [2((\int_0^t g(t-s) B_0(u, u_\xi)(s) ds, B_0(u, u_\xi)(t)\ ))_0 \\
& \qquad + ((\int_0^t g(t-s) B_0(u_\xi, u_\xi)(s) ds, B_0(u, u)(t)\ ))_0\ ] d\xi \le 0.
\end{aligned}
$$

After applying the growth assumption (37) we arrive at the inequality with an arbitrary $\epsilon > 0$:

$$(1-\epsilon)[\|u(t)\|^2 + 2\alpha \int_0^1 \|B_0(u(t),u_\xi(t))\|^2 d\xi] \leq \tag{82}$$
$$\alpha\|B_0\|^2(1+\frac{K}{\beta})\max\{\max_{t\in[0,T]}\|u_1(t)\|^2, \max_{t\in[0,T]}\|u_2(t)\|^2\} +$$
$$C(\epsilon)\int_0^t [\|u(s)\|^2 + 2\alpha\int_0^1 \|B_0(u(s),u_\xi(s))\|^2 d\xi]ds \quad \text{for all } t\in[0,T]$$

The inequality (82) holds for an arbitrary $\epsilon > 0$. If the solutions $u_1,\ u_2$ fulfil the estimate

$$\|u_i(t)\|^2 < [\alpha\|B_0\|^2(1+\frac{K}{\beta})]^{-1} \text{ for all } t\in[0,T],\ i=1,\ 2, \tag{83}$$

then there exists such $\epsilon > 0$ that there holds the inequality

$$[\|u(t)\|^2 + 2\alpha\int_0^1 \|B_0(u(t),u_\xi(t))\|^2 d\xi] \leq$$
$$\frac{C(\epsilon)}{\epsilon}\int_0^t [\|u(s)\|^2 + 2\alpha\int_0^1 \|B_0(u(s),u_\xi(s))\|^2 d\xi]ds \quad \text{for all } t\in[0,T]$$

and the uniqueness of a solution follows after applying the Gronwall lemma.

We shall verify that the bound (78) of the right-hand side q implies the estimate (83).

Using the estimate (51) we obtain from the identity (79) the inequality

$$\|u_i\|^2 + \alpha\|B_0(u_i,u_i)\|^2 +$$
$$((\ g*u_i(t),u_i(t)\ )) + \alpha((\ g*B_0(u_i,u_i)(t),B_0(u_i,u_i)(t)\ ))$$
$$\leq ((\ q(t),u_i(t)\ )) \text{ for all } t\in[0,T].$$

After applying the growth condition (37) we obtain in a standard way the estimate

$$\max_{0\in[0,T]}\|u_i(t)\| \leq (1-\frac{K}{\beta})^{-1}\max_{0\in[0,T]}\|q(t)\| \tag{84}$$

and the bound (78) then really implies the estimate (83) which verrifies the uniqueness of a solution w of the canonical Volterra integral equation (36). The expression (33) is equivalent with the identity (19) and the uniqueness of a weak solution $\{w,\Phi\} = \{w,\Psi+F\}$ follows.

Remark 3.3 *If we compare the estimates (59) and (78) we can see that in the case of constants* $\beta,\ K$ *fulfilling the relation*

$$\frac{\beta}{K} \geq \frac{5+\sqrt{17}}{2}$$

the condition (59) implies the condition (78) and the whole sequence $\{\bar{w}_n, \bar{\Psi}_n + F\}$ *is convergent to a unique weak solution* $\{w, \Phi\}$ *of the problem (1)-(6).*

In the case of an arbitrary $\beta > K$ *there exists a constant* $\gamma_0 \in (0, \beta - K)$ *such that for arbitrary* $\gamma \in (0, \gamma_0)$ *the condition (59) implies the uniqueness condition (78).*

References

[1] I. Bock, *On the generalized von Kármán system for the viscoelastic plate and its semidiscretization. I. Pseudoparabolic case*, submitted to Z. Angew. Math. Mech.

[2] T. Cazenave and A. Haraux, *An Introduction to Semilinear Evolution Equations*, Clarendon Press, Oxford 1998.

[3] R. M. Christensen, *Theory of Viscoelasticity*, Academic Press, New York, 1982.

[4] P. G. Ciarlet and P. Rabier, *Les Équations de von Kármán*, Springer Verlag, Berlin, 1980.

[5] I. Hlaváček and J. Naumann, *Inhomogeneous boundary value problems for the von Kármán equations I.* , Aplikace matematiky 19 (1974) No.4, pp. 253-270.

[6] M. A. Horn and I. Lasiecka, *Uniform decay of weak solutions to a von Kármán plates with nonlinear dissipation*, Differential and Integral Equations 7, (1994), pp. 885–908.

[7] J. Kačur, *Application of Rothe's method to integro-differential equations*, J. reine angew. Math. 388, (1988), pp. 73-105.

[8] J. E. Lagnese, *Boundary stabilization of thin plates*, SIAM, Philadelphia, 1989.

[9] E. Muñoz Rivera and G. Perla Menzala *Decay rates of solutions to a von Kármán system for viscoelastic plates with memory*, Quarterly of Applied Math., to appear.

[10] J. Nečas and I. Hlaváček *Mathematical theory of elastic and elastico-plastic bodies*, Elsevier, Amsterdam, 1981.

[11] M. Slodička, *Error estimates of approximate solutions for a quasilinear parabolic integro-differential equations*, Aplikace Math. 34 (1989), pp. 439-448.