Finite homogeneous and lattice ordered effect algebras

Gejza Jenča

Department of Mathematics
Faculty of Electrical Engineering and Information Technology
Slovak Technical University
Ilkovičova 3
812 19 Bratislava
Slovakia
jenca@kmat.elf.stuba.sk

1 Introduction

Effect algebras (or D-posets) have recently been introduced by Foulis and Bennett in [1] for study of foundations of quantum mechanics. (See also [2], [3].) The prototype effect algebra is $(\mathcal{E}(\mathbb{H}), \oplus, 0, I)$, where $\mathbb{H}$ is a Hilbert space and $\mathcal{E}(\mathbb{H})$ consists of all self-adjoint operators A of $\mathbb{H}$ such that $0 \leq A \leq I$. For $A, B \in \mathcal{E}(\mathbb{H})$, $A \oplus B$ is defined iff $A + B \leq 1$ and then $A \oplus B = A + B$. $\mathcal{E}(\mathbb{H})$ plays an important role in the foundations of quantum mechanics [4], [5].

The class of effect algebras includes orthoalgebras [6] and a subclass (called MV-effect algebras or Boolean D-posets or Boolean effect algebras), which is essentially equivalent to MV-algebras, introduced by Chang in [7] (see for example [8], [9] for results on MV-algebras in the context of effect algebras). The class of orthoalgebras includes other classes of well-known orthostructures, like orthomodular posets [10] and orthomodular lattices [11],[12].

One of the most important results in the theory of effect algebras was proved by Riečanová in her paper [13]. She proved that every lattice ordered effect algebra is a union of maximal mutually compatible sub-effect algebras, called blocks. This result generalizes the well-known fact that an orthomodular lattice is a union of its maximal Boolean subalgebras. Moreover, as proved in [14], in every lattice ordered effect algebra E the set of all sharp elements forms a sub-effect algebra E_S, which is a sub-lattice of E; E_S is then an orthomodular lattice, and every block of E_S is the center of some block of E.

In [15], a new class of effect algebras, called *homogeneous effect algebras* was introduced. The class of homogeneous effect algebras includes orthoalgebras, effect algebras satisfying the *Riesz decomposition property* (see for example [16], [17]) and lattice-ordered effect algebras. Every homogeneous effect algebra is a union of its blocks, which are the maximal effect algebras satisfying the Riesz decomposition property.

In the present paper, we show that every finite homogeneous effect algebra is a homomorphic image of a finite orthoalgebra an that every finite lattice ordered effect algebra is a homomorphic image of a finite orthomodular lattice. Moreover, the surjective homomorphism preserves blocks in both directions : the (pre)image of a block is always a block.

2 Definitions and basic relationships

An *effect algebra* is a partial algebra $(E;\oplus,0,1)$ with a binary partial operation $\oplus$ and two nullary operations $0,1$ satisfying the following conditions.

(E1) If $a\oplus b$ is defined, then $b\oplus a$ is defined and $a\oplus b=b\oplus a$.
(E2) If $a\oplus b$ and $(a\oplus b)\oplus c$ are defined, then $b\oplus c$ and $a\oplus(b\oplus c)$ are defined and $(a\oplus b)\oplus c=a\oplus(b\oplus c)$.
(E3) For every $a\in E$ there is a unique $a'\in E$ such that $a\oplus a'=1$.
(E4) If $a\oplus 1$ exists, then $a=0$

Effect algebras were introduced by Foulis and Bennett in their paper [1]. Independently, Kôpka and Chovanec introduced an essentially equivalent structure called *D-poset* (see [2]). Another equivalent structure, called *weak orthoalgebras* was introduced by Giuntini and Greuling in [3]. We refer to [18] for more information on effect algebras and D-posets.

For brevity, we denote the effect algebra $(E, \oplus, 0, 1)$ by E. In an effect algebra E, we write $a \leq b$ iff there is $c \in E$ such that $a \oplus c = b$. It is easy to check that every effect algebra is cancellative, thus $\leq$ is a partial order on E. In this partial order, 0 is the least and 1 is the greatest element of E. Moreover, it is possible to introduce a new partial operation $\ominus$; $b \ominus a$ is defined iff $a \leq b$ and then $a \oplus (b \ominus a) = b$. It can be proved that $a \oplus b$ is defined iff $a \leq b'$ iff $b \leq a'$. Therefore, it is usual to denote the domain of $\oplus$ by $\perp$. If $a \perp b$, we say that a and b are *orthogonal.* We write shortly

$$n.a := \overbrace{a \oplus \ldots \oplus a}^{n \text{ times}}.$$

We say that $\iota(a) = \max\{n : n.a \text{ exists}\}$ is *the isotropic index of a*. The isotropic index of a nonzero element may be infinite; however, in a finite effect algebra every nonzero element has a finite isotropic index. An element a on an effect algebra is *sharp* iff $a \wedge a' = 0$. The set of all sharp elements of an effect algebra E is denoted by E_S. An element a of an effect algebra is an *atom* iff $x < a \implies x = 0$. The set of all atoms of an effect algebra E is denoted by $\mathrm{At}(E)$.

An effect algebra need not be lattice ordered. However, if $x \perp y$ and $x \vee y$ exists then $x \wedge y$ exists and $x \oplus y = (x \wedge y) \oplus (x \vee y)$.

Let $E_0 \subseteq E$ be such that $1 \in E_0$ and, for all $a, b \in E_0$ with $a \geq b$, $a \ominus b \in E_0$. Since $a' = 1 \ominus a$ and $a \oplus b = (a' \ominus b)'$, E_0 is closed with respect to $\oplus$ and $'$. We then say that $(E_0, \oplus, 0, 1)$ is a *sub-effect algebra of E.*

Let E_1, E_2 be effect algebras. A map $\phi : E_1 \mapsto E_2$ is called a *homomorphism* iff it satisfies the following condition.

(H1) $\phi(1) = 1$ and if $a \perp b$, then $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

A homomorphism $\phi : E_1 \mapsto E_2$ of effect algebras is called *full* iff the following condition is satisfied.

(H2) If $\phi(a) \perp \phi(b)$ $\phi(a)\oplus\phi(b) \in \phi(E_1)$ then there exist $a_1, b_1 \in E_1$ such that $a_1 \perp b_1$, $\phi(a) = \phi(a_1)$ and $\phi(b) = \phi(b_1)$.

A bijective, full homomorphism is called an *isomorphism*.

Remark. For our purposes, it is natural to consider orthomodular lattices, orthomodular posets, MV-algebras, and Boolean algebras as special types of effect algebras. In the present paper, we will write shortly "orthomodular lattice" instead of "effect algebra associated with an orthomodular lattice" and similarly for orthomodular posets, MV-algebras, and Boolean algebras.

An effect algebra satisfying $a \perp a \implies a = 0$ is called an *orthoalgebra* (cf. [6]). It is easy to see that an effect algebra E is an orthoalgebra iff $E = E_S$. An orthoalgebra is an *orthomodular lattice* iff it is lattice ordered. A lattice ordered effect algebra is an *MV-algebra* iff $a \wedge b = 0$ implies that $a \perp b$. The class of *Boolean algebras* is the intersection of the classes of MV-algebras and orthomodular lattices. An effect algebra E satisfies *Riesz decomposition property* iff $u \leq v_1 \oplus \ldots \oplus v_n$ implies that there exist $u_1, \ldots, u_n \in E$ such that $u_i \leq v_i$ and $u = u_1 \oplus \ldots \oplus u_n$. An effect algebra satisfies Riesz decomposition property iff it satisfies Riesz decomposition property with fixed $n = 2$. An effect algebra is *homogeneous* iff $u \leq v_1 \oplus \ldots \oplus v_n \leq u'$ implies that there exist $u_1, \ldots, u_n \in E$ such that $u_i \leq v_i$ and $u = u_1 \oplus \ldots \oplus u_n$. An effect algebra is homogeneous iff iff it satisfies the above condition with fixed $n = 2$ (see [15], Proposition 2.3). The class of homogeneous effect algebras includes orthoalgebras, lattice ordered effect algebras and effect algebras satisfying Riesz decomposition property (see [15], Proposition 2.2).

Let E be an effect algebra. Let $C = (c_1, \ldots, c_n)$ be a n-tuple of elements of E. We say that C is *orthogonal word* iff the sum $c_1 \oplus \ldots \oplus c_n$ exists. We then write $\bigoplus C = c_1 \oplus \ldots \oplus c_n$. For $n = 0$, we put $\bigoplus C = 0$. We say that $Ran(C) = \{c_1, \ldots, c_n\}$ is *the range of* C.

A finite subset M_F of an effect algebra E is called *compatible with cover in* $X \subseteq E$ iff there is a finite orthogonal word $C = (c_1, \ldots, c_n)$ with $Ran(C) \subseteq X$ such that for every $a \in M_F$ there is a set $A \subseteq \{1, \ldots, n\}$ with $a = \bigoplus_{i \in A} c_i$. C is then called an *orthogonal cover* of M_F. A subset M of E is called *compatible with covers in* $X \subseteq E$ iff every finite subset of M is compatible with covers in X. A subset M of E is called *internally compatible* iff M is compatible with covers in M. A subset M of E is called *compatible* iff M is compatible with covers in E. An effect algebra E is said to be *compatible* if E is a compatible subset of E.

By [13], every maximal compatible subset of a lattice ordered effect algebra E is an MV-algebra, which is an sub-effect algebra of E and a sublattice of E. On the other hand, every MV-algebra is compatible. Thus, maximal compatible subsets of lattice ordered effect algebras coincide with maximal sub-effect algebras which are MV-algebras. Such sub-effect algebras are called *blocks*. In [15], these results were generalized for the class of homogeneous effect algebras. In a homogeneous effect algebra, the blocks are maximal effect algebras satisfying Riesz decomposition property. Similarly as in the lattice ordered case, the blocks in the homogeneous case are exactly the maximal internally compatible subsets. In the case of a finite homogeneous effect algebra, every maximal compatible subset is internally compatible (this follows from the results in [15]), so the peculiarities of internally compatible sets are not important – the block coincide with maximal compatible sets. Moreover, since every finite effect algebra satisfying Riesz decomposition property is an MV-algebra (see [18]) and since every finite MV-algebra is a direct product of totally ordered finite MV-algebras, every block of a finite homogeneous effect algebra is a direct product of totally ordered MV-algebras.

In every homogeneous effect algebra, the set of all sharp elements forms an sub-effect algebra, which is an orthoalgebra. (see [15], Section 4). Moreover in a lattice ordered effect algebra E, E_S is an orthomodular lattice, which is a sublattice of E (see [14]).

An *E-test space* is a pair $(X, \mathcal{T})$, where X is a nonempty set and $\mathcal{T} \subseteq \mathbb{N}_0^X$ ($\mathbb{N}_0 := \{0, 1, 2, \ldots\}$), where the following conditions are satisfied.

(T1) For all $x \in X$ there exists $\mathbf{t} \in \mathcal{T}$ such that $\mathbf{t}(x) > 0$.
(T2) For all $\mathbf{s}, \mathbf{t} \in \mathcal{T}$, $\mathbf{s} \leq \mathbf{t}$ implies that $\mathbf{s} = \mathbf{t}$.

The elements of $\mathcal{T}$ are called *tests* of $(X, \mathcal{T})$ and the elements of X are called *outcomes*.

E-test spaces in this form were introduced by Gudder in [19]. In [18], an essentially equivalent notion of *D-test space* was introduced and studied. Both definitions generalize *test spaces* (see [20] or [21]).

Let $(X, \mathcal{T})$ be an E-test space. We say that a mapping $\mathbf{f} \in \mathbb{N}_0^X$ is an *event* of $(X, \mathcal{T})$ iff there is a test $\mathbf{t}$ such that $\mathbf{t} \geq \mathbf{f}$. We say that two events $\mathbf{f}$ and $\mathbf{g}$ are

(i) *orthogonal*, in symbols $\mathbf{f} \perp \mathbf{g}$, iff $\mathbf{f} + \mathbf{g}$ is an event;
(ii) *local complements*, in symbols $\mathbf{f}$ loc $\mathbf{g}$ iff $\mathbf{f} + \mathbf{g}$ is a test;
(iii) *perspective*, in symbols $\mathbf{f} \sim \mathbf{g}$, iff they share a common local complement $\mathbf{h}$.

An E-test space $(X, \mathcal{T})$ is algebraic iff, for all events $\mathbf{f}, \mathbf{g}, \mathbf{h}$, $\mathbf{f} \sim \mathbf{g}$, $\mathbf{g}$ loc $\mathbf{h}$ imply $\mathbf{f} \perp \mathbf{h}$. For every algebraic E-test space $(X, \mathcal{T})$, $\sim$ is an equivalence relation on the set of all events. Moreover, the set of all equivalence classes of events can be organized into an effect algebra with a partial binary operation $\oplus$ defined as follows: if $[\mathbf{f}]_\sim [\mathbf{g}]_\sim$ are equivalence classes of events, then $[\mathbf{f}]_\sim \oplus [\mathbf{g}]_\sim$ exists iff $\mathbf{f} \perp \mathbf{g}$ and then $[\mathbf{f}]_\sim \oplus [\mathbf{g}]_\sim = [\mathbf{f} + \mathbf{g}]_\sim$. The unit element is the equivalence class of all tests and the zero element is the zero constant map. This effect algebra is called *the effect algebra of*

a	b
3	0
2	1

Fig. 1. A six element effect algebra and its atomic test space

$(X, \mathcal{T})$. It can be proved that every effect algebra E arises as an effect algebra of a suitable E-test space $(X, \mathcal{T})$, called *the E-test space of* E, which is constructed as follows: $X = E$ and $\mathcal{T}$ is the set of all mappings $\mathbf{f} \in \mathbb{N}_0^E$ such that $\mathrm{supp}(\mathbf{f})$ is a finite set and $\bigoplus_{a \in \mathrm{At}(E)} \mathbf{f}(a).a$ exists and equals 1. The tests of the test space of E are called *tests of* E.

Let E be a finite effect algebra. The *atomic E-test space of* E is the pair $(\mathrm{At}(E), \mathcal{T})$, where $\mathrm{At}(E)$ is the set of all atoms of E and $\mathcal{T}$ is the set of all tests $\mathbf{t}$ of E such that $\mathrm{supp}(\mathbf{t}) \subseteq \mathrm{At}(E)$.

The events of an atomic E-test space of a finite effect algebra E are called atomic events of E. For an atomic event $\mathbf{f}$ of E, we write shortly $\bigoplus \mathbf{f}$ instead of $\bigoplus_{a \in \mathrm{At}(E)} \mathbf{f}(a).a$. Whenever $x \in E$ and $\bigoplus \mathbf{f} = x$, we say that $\mathbf{f}$ is an atomic decomposition of x. It is easy to check that every atomic E-test space is algebraic and that the effect algebra of the atomic test space of a finite effect algebra E is isomorphic to E. Thus, a finite effect algebra is determined by its atomic test space.

Example 2.1 *Let* $E = \{0, a, b, c, d, 1\}$ *be a partial algebra satisfying* $\forall x : x \oplus 0 = 0 \oplus x = x$, $a \oplus a = b \oplus b = c$, $a \oplus b = b \oplus a = d$ $c \oplus a = a \oplus c = 1$, $d \oplus b = b \oplus d = 1$. *In all other cases,* $\oplus$ *is undefined. Then* E *is an effect algebra. We have* $At(E) = \{a, b\}$ *and the atomic test space of* E *has two tests. (see Figure 2.1) This is the most simple example of a non-lattice ordered effect algebra.*

3 Atomic E-test spaces of effect algebras and their sharpenings

In this section, we prove that for every finite homogeneous effect algebra E there is an orthoalgebra $O(E)$ and a full homomorphism $\phi_E : O(E) \to E$. Moreover, both ϕ_E and ϕ_E^{-1} preserve blocks so the structure of $O(E)$ is essentially the same as the structure of E. Let us start with a useful characterization of finite homogeneous effect algebras in terms of their atomic E-test spaces.

Proposition 3.1 *For every finite effect algebra E, the following are equivalent.*

(a) E *is homogeneous.*
(b) Let $\mathbf{u}, \mathbf{f}$ *be a pair of atomic events such that* $\oplus \mathbf{u} \leq \oplus \mathbf{f} \leq (\oplus \mathbf{u})'$. *Then* $\mathbf{u} \leq \mathbf{f}$.
(c) For every atom a *and for every atomic event* $\mathbf{f}$ *such that* $a \leq \oplus \mathbf{f} \leq a'$, $a \in \mathrm{supp}(\mathbf{f})$.
(d) Let $\mathbf{f}, \mathbf{g}$ *be atomic tests, let* $a \in \mathrm{supp}(\mathbf{f}) \cap \mathrm{supp}(\mathbf{g})$. *Then* $\mathbf{f}(a) = \mathbf{g}(a)$.
(e) For every atom a *and every atomic event* $\mathbf{f}$ *such that* $a \in \mathrm{supp}(\mathbf{f})$, $\mathbf{f}(a) = \iota(a)$.

PROOF. (d) $\implies$ (c): Since $\oplus \mathbf{f} \leq a'$, there is an atomic event $\mathbf{g}$ such that $\mathbf{f}$ loc $\mathbf{g}$ and $a \in \mathrm{supp}(\mathbf{g})$. Similarly, since $a \leq \oplus \mathbf{f}$, there is an atomic decomposition $\mathbf{h}$ of $\oplus \mathbf{f}$ such that $a \in \mathrm{supp}(\mathbf{h})$. Since both $\mathbf{f}+\mathbf{g}$ and $\mathbf{h}+\mathbf{g}$ are atomic tests and $a \in \mathrm{supp}(\mathbf{f}+\mathbf{g}) \cap \mathrm{supp}(\mathbf{h}+\mathbf{g})$, by (d) we obtain $(\mathbf{f}+\mathbf{g})(a) = (\mathbf{h}+\mathbf{g})(a)$. Hence $\mathbf{f}(a) = \mathbf{h}(a) \geq 1$ and we see that $a \in \mathrm{supp}(\mathbf{f})$

(c) $\implies$ (b): Let us write $|\mathbf{u}| = \sum_{a \in \mathrm{At}(E)} \mathbf{u}(a)$. For $|\mathbf{u}| = 0$ we have $\mathbf{u} = 0$ and there is nothing to prove. Suppose that (b) is valid for all $\mathbf{u}$ such that $|\mathbf{u}| = n$. Let $\mathbf{u}$ be an atomic event with $|\mathbf{u}| = n+1$. Pick $a \in \mathrm{supp}(\mathbf{u})$. Let us denote the characteristic function of $\{a\}$ by χ_a. Since

$$a \leq \bigoplus \mathbf{u} \leq \bigoplus \mathbf{f} \leq (\bigoplus \mathbf{u})' \leq a',$$

the assumptions of (c) are satisfied. Hence, $a \in \mathrm{supp}(\mathbf{f})$ and $\chi_a \leq \mathbf{f}$. Obviously,

$$\bigoplus(\mathbf{u} - \chi_a) \leq \bigoplus(\mathbf{f} - \chi_a) \leq \bigoplus \mathbf{f} \leq (\bigoplus \mathbf{u})' \leq \bigoplus(\mathbf{u} \ominus \chi_a)'.$$

Since $|\mathbf{u} - \chi_a| = n$, we may apply the induction hypothesis to obtain $\mathbf{u} - \chi_a \leq \mathbf{f} - \chi_a$. This implies that $\mathbf{u} \leq \mathbf{f}$.

(b) $\implies$ (a): Let u, v_1, v_2 be such that $u \leq v_1 \oplus v_2 \leq u'$. Let $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2$ be atomic decompositions of u, v_1, v_2, respectively. By (b), $\mathbf{u} \leq \mathbf{v}_1 + \mathbf{v}_2$. It is easy to check that there exist atomic events $\mathbf{u}_1, \mathbf{u}_2$ such that $\mathbf{u}_1 \leq \mathbf{v}_1$, $\mathbf{u}_2 \leq \mathbf{v}_2$ and $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$.

(a) $\implies$ (d): Suppose $\mathbf{f}(a) < \mathbf{g}(a)$. Let $(v_1, \dots, v_n)$ be a sequence of all atoms from $\mathrm{supp}(\mathbf{f}) \setminus \{a\}$, such that every atom b occurs in the sequence $\mathbf{f}(b)$ times. Then $v_1 \oplus \dots \oplus v_n$ exists and $v_1 \oplus \dots \oplus v_n \leq a'$. Moreover, since $\mathbf{g}(a).a \leq 1 = \mathbf{f}(a).a \oplus v_1 \oplus \dots \oplus v_n$, we obtain $a \leq (\mathbf{g}(a) - \mathbf{f}(a)).a \leq v_1 \oplus \dots \oplus v_n$. Since E is homogeneous, there exist $a_1, \dots, a_n$ such that, $a_i \leq v_i$ and $a = a_1 \oplus \dots \oplus a_n$. Since $a, v_1, \dots, v_n$ are atoms, this implies that $a = a_i = v_i$ for some i. This contradicts the assumption $a \notin \{v_1, \dots, v_n\}$.

The proof of the equivalence of (d) and (e) is left to the reader.

Corollary 3.2 *Let E be homogeneous effect algebra, let $\mathbf{w}$ be an atomic event of E such that $\oplus \mathbf{w} \in E_S$. For every $a \in \mathrm{supp}(\mathbf{w})$, $\mathbf{w}(a) = \iota(a)$.*

PROOF. Suppose that $\mathbf{w}(a) < \iota(a)$ Let $\mathbf{t}$ be an atomic test such that $\mathbf{w} \leq \mathbf{t}$. By Proposition 3.1, $\mathbf{t}(a) = \iota(a)$. Therefore $(\mathbf{t} - \mathbf{w})(a) \geq 1$ and $a \leq \oplus(\mathbf{t} - \mathbf{w}) \perp \oplus \mathbf{w}$. Since $a \in \mathrm{supp}(\mathbf{w})$, $a \leq \oplus \mathbf{w}$. This contradicts $\oplus \mathbf{w} \in E_S$.

By Proposition 3.1, it is easy to see that the effect algebra from Example 2.1 is not homogeneous.

Proposition 3.3 *Let E be finite homogeneous effect algebra, let $\mathbf{t}$ be an atomic test of E. Then*

$$B = \{\bigoplus \mathbf{f} : \mathbf{f} \leq \mathbf{t}\} \tag{1}$$

is a block of of E. Moreover, for every block B there is a unique atomic test $\mathbf{t}$ satisfying (1).

PROOF. Clearly, B is a finite compatible set. By [15], Corollary 3.12, there is a block $B_0 \supseteq B$. Suppose that $B_0 \neq B$ and let $a \in B_0 \backslash B$. Since $a \leq \oplus \mathbf{t}$ and B_0 satisfies the Riesz decomposition property, $a \in \text{supp}(\mathbf{t})$. This contradicts $a \notin B$, so $B = B_0$.

Conversely, let B be a block of E. Let $\mathbf{t}_1, \mathbf{t}_2$ be atomic tests of B. If $\text{supp}(\mathbf{t}_1) = \text{supp}(\mathbf{t}_2)$ then, by Proposition 3.1, $\mathbf{t}_1 = \mathbf{t}_2$. Let $a \in \text{supp}(\mathbf{t}_1)$, $a \notin \text{supp}(\mathbf{t}_2)$. Similarly as above, $a \leq \oplus \mathbf{t}_2 = 1$ implies that $a \in \text{supp}(\mathbf{t}_2)$. This is a contradiction. Thus, B has a unique atomic test $\mathbf{t}$. It remains to prove that this $\mathbf{t}$ is an atomic test of E. Let a be an atom of B, let $x \in E$, $0 < x \leq a$. Let $C = (c_1, \dots, c_n)$ orthogonal cover of B. Since $1 \in B$, $\oplus C = 1$. Since a is an atom of B, $a = c_i$ for some i, say $i = 1$. Then $C_0 = (x, a \ominus x, c_2, \dots, c_n)$ is an orthogonal word, which covers $B \cup \{x\}$. Hence $B \cup \{x\}$ is a finite compatible set and, by the maximality of B, $x \in B$. Since a is an atom of B, $x = a$. Thus, every atom of B is an atom of E and $\mathbf{t}$ is a unique atomic test of E satisfying (1).

Example 3.4 *Let E be an effect algebra with the atomic test space given by the following table*

a	b	c	d	e	f
1	1	1	0	0	0
0	0	1	2	1	0
1	0	0	0	1	1

By Proposition 3.1, E is homogeneous. By Proposition 3.3, E has *three blocks. We remark that E is not lattice ordered. The Hasse diagram of E can be found in [15].*

Let $(X, \mathcal{T})$ be an E-test space. Let us construct another pair $(X_S, \mathcal{T}_S)$, where $\mathcal{T}_S \subseteq \mathbb{N}_0^{X_S}$, called *the sharpening of* $(X, \mathcal{T})$. The outcome space $X_S \subseteq X \times \mathbb{N}$ is given by

$$(x, n) \in X_S \text{ iff } \exists \mathbf{t} \in \mathcal{T} : \mathbf{t}(x) \geq n.$$

Every $\mathbf{t}_S \in \mathcal{T}_S$ is constructed from a test $\mathbf{t} \in \mathcal{T}$ by

$$\mathbf{t}_S(x, n) = \begin{cases} 1 & \text{if } \mathbf{t}(x) \geq n \\ 0 & \text{otherwise} \end{cases}$$

Note that every $\mathbf{t}_S$ is just a characteristic function of its support. In what follows, we simply identify $\mathbf{t}_S$ with its support, so that e.g. $(x, n) \in \mathbf{t}_S$ means that $\mathbf{t}_S(x, n) = 1$ and $\mathbf{t}_S \subseteq \mathbf{r}_S$ means that $\mathbf{t}_S \leq \mathbf{r}_S$. Since, for all $\mathbf{t} \in \mathcal{T}$ and for all $a \in X$,

$$\mathbf{t}(a) = \max\{n : (a, n) \in \mathbf{t}_S\},$$

the map $\mathbf{t} \mapsto \mathbf{t}_S$ is a bijection.

Proposition 3.5 *The sharpening of an E-test space is an E-test space.*

PROOF. Let $(X, \mathcal{T})$ be an E-test space, let $(X_S, \mathcal{T}_S)$ be its sharpening. Let $\mathbf{t}_S, \mathbf{r}_S \in \mathcal{T}_S$ be such that $\mathbf{t}_S \subseteq \mathbf{r}_S$. Let $\mathbf{t}, \mathbf{r} \in \mathcal{T}$ be tests corresponding to $\mathbf{t}_S, \mathbf{r}_S$, respectively. For all $a \in X$,

$$\mathbf{t}(a) = \max\{n : (a, n) \in \mathbf{t}_S\} \leq \max\{n : (a, n) \in \mathbf{r}_S\} = \mathbf{r}(a).$$

Thus, $\mathbf{t} \leq \mathbf{r}$. Since $(X, \mathcal{T})$ is an E-test space, $\mathbf{t} \leq \mathbf{r}$ implies that $\mathbf{t} = \mathbf{r}$. Therefore, $\mathbf{t}_S = \mathbf{r}_S$.

The following example shows that a sharpening of an algebraic E-test space need not be algebraic.

Example 3.6 *Let C_{10} be an effect algebra with a single atom a such that $10a = 1$. Let $\mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3 \in \mathbb{N}_0^{C_{10}}$ be tests of the E-test space*

of C_{10} *given by*
$\mathrm{supp}(\mathbf{t}^1) = \{2a, 6a\}$, $\mathbf{t}^1(2a) = 2$, $\mathbf{t}^1(6a) = 1$
$\mathrm{supp}(\mathbf{t}^2) = \{2a, 4a\}$, $\mathbf{t}^2(2a) = 1$, $\mathbf{t}^2(4a) = 2$
$\mathrm{supp}(\mathbf{t}^3) = \{1a, 4a\}$, $\mathbf{t}^3(1a) = 2$, $\mathbf{t}^3(4a) = 2$.
The corresponding tests of the sharpened E-test space are
$\mathbf{t}^1_S = \{(2a, 1), (2a, 2), (6a, 1)\}$
$\mathbf{t}^2_S = \{(2a, 1), (4a, 1), (4a, 2)\}$
$\mathbf{t}^3_S = \{(1a, 1), (1a, 2), (4a, 1), (4a, 2)\}$.
In the sharpened E-test space we have

$$\{(2a, 2), (6a, 1)\} \sim \{(4a, 1), (4a, 2)\} \textit{ loc } \{(1a, 1), (1a, 2)\}.$$

However, since $\{(2a, 2), (6a, 1), (1a, 1), (1a, 2)\}$ *is not a test,* $\{(2a, 2), (6a, 1)\}$ *and* $\{(1a, 1), (1a, 2)\}$ *are not local complements of each other.*

Let E be a finite effect algebra, let $(\mathrm{At}(E)_S, \mathcal{T}_S)$ be the sharpening of the atomic E-test space of E. We write shortly $\Omega(E) = (\mathrm{At}(E)_S, \mathcal{T}_S)$. For an event $\mathbf{f} = \{(a_1, n_1), \ldots, (a_k, n_k)\}$ of $\Omega(E)$, we write shortly $\bigoplus \mathbf{f} = a_1 \oplus \ldots \oplus a_k$. Let $\mathbf{t}$ be an event of $\Omega(E)$, $a \in \mathrm{At}(E)$. We say that a occurs in $\mathbf{t}$ iff there exists $n \in \mathbb{N}$ such that $(a, n) \in \mathbf{t}$.

Lemma 3.7 *Let* E *be a finite homogeneous effect algebra. For all events* $\mathbf{f}, \mathbf{g}$ *of* $\Omega(E)$ *we have* $\mathbf{f} \perp \mathbf{g}$ *iff* $\mathbf{f} \cap \mathbf{g} = \emptyset$ *and* $\bigoplus \mathbf{f} \perp \bigoplus \mathbf{g}$.

PROOF. Let $\mathbf{h} \in \mathbb{N}_0^{\mathrm{At}(E)}$ be given by

$$\mathbf{h}(a) = |\{(a, n) : (a, n) \in \mathbf{f} \dot{\cup} \mathbf{g}\}|.$$

Since $\bigoplus \mathbf{h}$ exists, there is an atomic test $\mathbf{t} \geq \mathbf{h}$. Let us prove that $\mathbf{f} \dot{\cup} \mathbf{g} \subseteq \mathbf{t}_S$. Suppose $(a, n) \in \mathbf{f} \dot{\cup} \mathbf{g}$, $n \in \mathbb{N}$. Without loss of generality, we may assume $(a, n) \in \mathbf{f}$. Since $\mathbf{f}$ is an event of $\Omega(E)$, there exists an atomic test $\mathbf{r}$ such that $\mathbf{r}(a) \geq n$. Since, by Proposition 3.1 part (d), $\mathbf{r}(a) = \mathbf{t}(a)$, we have $\mathbf{t}(a) \geq n$, which means that $(a, n) \in \mathbf{t}_S$. Similarly, $(a, n) \in \mathbf{g}$ implies that $(a, n) \in \mathbf{t}_S$. Thus, $\mathbf{f} \dot{\cup} \mathbf{g} \subseteq \mathbf{t}_S$, which means that $\mathbf{f} \perp \mathbf{g}$.

The opposite implication follows by definition of $\Omega(E)$.

Lemma 3.8 *Let E be a finite homogeneous effect algebra. For all events $\mathbf{f}, \mathbf{g}$ of $\Omega(E)$, we have $\mathbf{f}$ loc $\mathbf{g}$ iff $\mathbf{f} \cap \mathbf{g} = \emptyset$, $\bigoplus\mathbf{f} \perp \bigoplus\mathbf{g}$ and $\bigoplus\mathbf{f} \oplus \bigoplus\mathbf{g} = 1$.*

PROOF. By Lemma 3.7.

Lemma 3.9 *Let E be a finite homogeneous effect algebra. For all events $\mathbf{f}, \mathbf{g}$ of $\Omega(E)$ such that $\bigoplus\mathbf{f} \in E_S$, we have $\mathbf{f} \perp \mathbf{g}$ iff $\bigoplus\mathbf{f} \perp \bigoplus\mathbf{g}$.*

PROOF. Let $\bigoplus\mathbf{f} \in E_S$, $\bigoplus\mathbf{f} \perp \bigoplus\mathbf{g}$. Suppose $(a, n) \in \mathbf{f} \cap \mathbf{g}$. Then $a \leq \bigoplus\mathbf{g}$ and $a \leq \bigoplus\mathbf{g} \leq (\bigoplus\mathbf{f})'$, which contradicts $\bigoplus\mathbf{f} \in E_S$. Therefore, $\mathbf{f} \cap \mathbf{g} = \emptyset$. By Lemma 3.7, $\mathbf{f} \perp \mathbf{g}$.

Again, the opposite implication follows by definition of $\Omega(E)$.

Proposition 3.10 *For every finite homogeneous effect algebra E, $\Omega(E)$ is an algebraic E-test space.*

PROOF. Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be events of $\Omega(E)$ such that $\mathbf{f} \sim \mathbf{g}$ and $\mathbf{f}$ loc $\mathbf{h}$. We shall prove that $\mathbf{g}$ loc $\mathbf{h}$. Since $\mathbf{f} \sim \mathbf{g}$, there is an event $\mathbf{u}$ such that $\mathbf{f}$ loc $\mathbf{u}$ and $\mathbf{g}$ loc $\mathbf{u}$. By Lemma 3.8, this implies $\mathbf{f} \cap \mathbf{u} = \mathbf{g} \cap \mathbf{u} = \emptyset$, $\bigoplus\mathbf{f} \oplus \bigoplus\mathbf{u} = \bigoplus\mathbf{g} \oplus \bigoplus\mathbf{u} = 1$. This implies that $\bigoplus\mathbf{f} = \bigoplus\mathbf{g}$. Similarly, since $\mathbf{g}$ loc $\mathbf{h}$ and $\mathbf{g}$ loc $\mathbf{u}$, $\bigoplus\mathbf{h} = \bigoplus\mathbf{u}$. Therefore, $\bigoplus\mathbf{g} = \bigoplus\mathbf{f} \perp \bigoplus\mathbf{u} = \bigoplus\mathbf{h}$.

Note that, by Proposition 3.1 part(e), for every atom a of E occurring in a test $\mathbf{t}_S$ of $\Omega(E)$, we have $\{(a, 1), \ldots, (a, \iota(a))\} \subseteq \mathbf{t}_S$.

Let us prove that $\mathbf{g} \cap \mathbf{h} = \emptyset$. Assume the contrary and let $(a, n) \in \mathbf{g} \cap \mathbf{h}$. Since $\mathbf{g}$ loc $\mathbf{u}$, $(a, n) \notin \mathbf{u}$. Suppose that a occurs in $\mathbf{f}$. Since a occurs in $\mathbf{f} \dot\cup \mathbf{u}$, $(a, n) \notin \mathbf{u}$ implies $(a, n) \in \mathbf{f}$. This contradicts $\mathbf{f}$ loc $\mathbf{h}$. Suppose that a does not occur in $\mathbf{f}$. Since a occurs in $\mathbf{f} \dot\cup \mathbf{h}$, $\{(a, 1), \ldots, (a, \iota(a))\} \subseteq \mathbf{h}$. We have

$$\iota(a).a \leq \bigoplus \mathbf{h} = \bigoplus \mathbf{u} \perp \bigoplus \mathbf{g} \geq a.$$

This implies that $a \perp \iota(a).a$, which is impossible.

We have $\mathbf{g} \cap \mathbf{h} = \emptyset$, $\oplus \mathbf{g} \perp \oplus \mathbf{h}$ and $\oplus \mathbf{g} \oplus \oplus \mathbf{h} = \oplus \mathbf{f} \oplus \oplus \mathbf{h} = 1$. By Lemma 3.8, this implies that $\mathbf{g}$ loc $\mathbf{h}$.

Proposition 3.11 *Let E be a finite effect algebra such that $\Omega(E)$ is algebraic. Let $O(E)$ be the effect algebra of $\Omega(E)$. Then $O(E)$ is an orthoalgebra and the mapping $\phi_E : O(E) \to E$ given by $\phi_E([\mathbf{f}]_\sim) = \oplus \mathbf{f}$ is a surjective full homomorphism.*

PROOF. Let $\mathbf{f}$ be an event of $\Omega(E)$. Suppose that $[\mathbf{f}]_\sim \perp [\mathbf{f}]_\sim$ in $O(E)$. Since $\Omega(E)$ is algebraic, $\mathbf{f} \perp \mathbf{f}$ in $\Omega(E)$. By the very definition of $\Omega(E)$, this implies that $\mathbf{f} = 0$. Therefore, $[\mathbf{f}]_\sim = 0$ and $O(E)$ is an orthoalgebra.

For all events $\mathbf{f}, \mathbf{g}$ of $\Omega(E)$, $\mathbf{f} \sim \mathbf{g}$ implies that $\oplus \mathbf{f} = \oplus \mathbf{g}$. Thus, ϕ_E is well defined. Since every element of E has an atomic decomposition, ϕ_E is surjective. Obviously, $\phi_E(1_{O(E)}) = 1$, since $1_{O(E)}$ is the set of all tests of $\Omega(E)$. Let $\mathbf{f}, \mathbf{g}$ be events of $\Omega(E)$, let $\mathbf{f} \perp \mathbf{g}$. Then there is a test $\mathbf{t}$ of $\Omega(E)$ such that $\mathbf{f} \dot\cup \mathbf{g} \subseteq \mathbf{t}$. Thus $\oplus \mathbf{f} \perp \oplus \mathbf{g}$ and

$$\phi_E([\mathbf{f}]_\sim) \perp \phi_E([\mathbf{g}]_\sim) = \bigoplus \mathbf{f} \oplus \bigoplus \mathbf{g} = \bigoplus (\mathbf{f} \dot\cup \mathbf{g}) = \phi_E([\mathbf{f}]_\sim \oplus [\mathbf{g}]_\sim).$$

It the remainder of this paper, we adopt the notations $O(E)$ and ϕ_E from above Proposition.

Theorem 3.12 *For every finite homogeneous effect algebra E, there is an orthoalgebra $O(E)$ and a surjective full homomorphism $\phi_E : O(E) \to E$.*

PROOF. Most of this follows by Propositions 3.10 and 3.11. It remains to prove that ϕ_E is full.

Let $[\mathbf{f}]_\sim, [\mathbf{g}]_\sim \in O(E)$ be such that $\phi_E([\mathbf{f}]_\sim) \perp \phi_E([\mathbf{g}]_\sim)$, that means, $\oplus \mathbf{f} \perp \oplus \mathbf{g}$. We shall proceed my induction with respect to $|\mathbf{f} \cap \mathbf{g}|$. Suppose $|\mathbf{f} \cap \mathbf{g}| = 0$. By Lemma 3.7, $\mathbf{f} \perp \mathbf{g}$. Suppose

$|\mathbf{f} \cap \mathbf{g}| = n+1$, $n \in \mathbb{N}_0$, let $(a,n) \in \mathbf{f} \cap \mathbf{g}$. Put $n_{\mathbf{f}} = |\{k \in \mathbb{N} : (a,k) \in \mathbf{f}|$ and $n_{\mathbf{g}} = |\{k \in \mathbb{N} : (a,k) \in \mathbf{g}|$. Since $\oplus \mathbf{f} \perp \oplus \mathbf{g}$, $n_{\mathbf{f}} + n_{\mathbf{g}} \leq \iota(a)$. Therefore, there exists $l \leq \iota(a)$ such that $(a,l) \notin \mathbf{f} \cup \mathbf{g}$. Put $\mathbf{f}_1 = (\mathbf{f} \setminus \{(a,n)\}) \dot{\cup} \{(a,l)\}$. We have $\phi([\mathbf{f}_1]_\sim) = \phi_([\mathbf{f}]_\sim)$ and $|\mathbf{f}_1 \cap \mathbf{g}| = n$, hence we may apply the induction hypothesis.

The following example shows that there are exists a non-homogeneous effect algebra E with algebraic $\Omega(E)$.

Example 3.13 *Let E be the effect algebra from Example 2.1. The outcome space of $\Omega(E)$ is*

$$\{(a,1),(a,2),(a,3),(b,1),(b,2)\}.$$

The tests of $\Omega(E)$ are given by the following table.

$(a,1)$	$(a,2)$	$(a,3)$	$(b,1)$	$(b,2)$
1	1	1	0	0
1	0	0	1	1

It is easy to see that $\Omega(E)$ is algebraic. $O(E)$ is an orthomodular lattice with two 8-elements blocks, the blocks of $O(E)$ share one of their atoms.

4 Properties of $O(E)$ and ϕ_E

In this section, we focus on finite homogeneous effect algebras. We introduce two arrow operators defined of the set of all events of $\Omega(E)$. We use them to characterize the perspectivity of events and the partial order of $O(E)$. We prove that the blocks structure of $O(E)$ is essentially the same as the blocks structure of E. Finally, prove that for a finite lattice ordered effect algebra E, $O(E)$ is a lattice.

Let E be a finite homogeneous effect algebra. For every event $\mathbf{u}$

of $\Omega(E)$, we write

$$\mathbf{u}^{\uparrow} = \bigcup_{(a,n)\in u} \{(a,1),\dots,(a,\iota(a))\}.$$

By Proposition 3.1, for every test $\mathbf{t}$ of $\Omega(E)$, $\mathbf{t} \supseteq \mathbf{u}$ implies that $\mathbf{t} \supseteq \mathbf{u}^{\uparrow}$. Hence $\mathbf{u}^{\uparrow}$ is an event. Dually, we write

$$\mathbf{u}^{\downarrow} = \bigcup_{\mathbf{v}\subseteq\mathbf{u},\ \mathbf{v}=\mathbf{v}^{\uparrow}} \mathbf{v}.$$

By definition, $\mathbf{u}^{\downarrow}$ is the greatest subevent of $\mathbf{u}$ satisfying $(a,n) \in \{^{\downarrow}\mathbf{u}\} \implies \{(a,1),\dots,(a,\iota(a))\} \subseteq \mathbf{u}^{\downarrow}$. Note that, by Proposition 3.1, for every test $\mathbf{t}$ of $\Omega(E)$ we have $\mathbf{t} = \mathbf{t}^{\uparrow}$.

Lemma 4.1 *Let E be a finite homogeneous effect algebra, let $a \in At(E)$, let $\mathbf{f}$ be a test of $\Omega(E)$. The following are equivalent.*

(a) a occurs in $\mathbf{f}^{\uparrow} \setminus \mathbf{f}^{\downarrow}$
(b) a occurs in $\mathbf{f} \setminus \mathbf{f}^{\downarrow}$
(c) a occurs in $\mathbf{f}^{\uparrow} \setminus \mathbf{f}$

PROOF.

(a)⇒(b): a occurs in $\mathbf{f}^{\uparrow}$ iff a occurs in $\mathbf{f}$. (b)⇒(c): a occurs in $\mathbf{f} \setminus \mathbf{f}^{\downarrow}$ iff there in $n \leq \iota(a)$ such that $(a,n) \notin \mathbf{f}$. For every such n, $(a,n) \in \mathbf{f}^{\uparrow} \setminus \mathbf{f}$. (c)⇒(a): $\mathbf{f}^{\uparrow} \setminus \mathbf{f} \subseteq \mathbf{f}^{\uparrow} \setminus \mathbf{f}^{\downarrow}$.

Proposition 4.2 *Let E be a finite homogeneous effect algebra, let $\mathbf{u}$ be an event of $\Omega(E)$. $\bigoplus\mathbf{u}$ is sharp iff $\mathbf{u} = \mathbf{u}^{\downarrow} = \mathbf{u}^{\uparrow}$*

PROOF. Note that $\mathbf{u} = \mathbf{u}^{\downarrow}$ iff $\mathbf{u} = \mathbf{u}^{\uparrow}$. Suppose that $\bigoplus\mathbf{u}$ is sharp and $\mathbf{u} \neq \mathbf{u}^{\uparrow}$. Let $(a,k) \in \mathbf{u}^{\uparrow} \setminus \mathbf{u}$, so that $a \leq (\bigoplus\mathbf{u})'$. By definition of $\mathbf{u}^{\uparrow}$, a occurs in $\mathbf{u}$, hence $a \leq \bigoplus\mathbf{u}$. This is a contradiction with the sharpness of $\bigoplus\mathbf{u}$.

Suppose that $\mathbf{u} = \mathbf{u}^{\uparrow}$ and that $\bigoplus\mathbf{u}$ is not sharp. There is an atom a such that $a \leq \bigoplus\mathbf{u}, (\bigoplus\mathbf{u})'$. This is equivalent to $a \leq \bigoplus\mathbf{u} \leq a'$. By the homogeneity of E, a occurs in $\mathbf{u}$. Since $\mathbf{u} = \mathbf{u}^{\uparrow}$ and a

occurs in $\mathbf{u}$, $\{(a,1),\dots,(a,\iota(a))\} \subseteq \mathbf{u}$. Therefore, $\iota(a).a \leq \bigoplus \mathbf{u} \perp a$. This is a contradiction.

Proposition 4.3 *Let E be a homogeneous effect algebra, let $\mathbf{u}, \mathbf{v}$ be events of $\Omega(E)$ such that $\bigoplus \mathbf{u} = \bigoplus \mathbf{v}$. Then $\bigoplus \mathbf{u}^{\downarrow} = \bigoplus \mathbf{v}^{\downarrow}$ and every atom a which occurs in $\mathbf{u} \setminus \mathbf{u}^{\downarrow}$ occurs in $\mathbf{u} \setminus \mathbf{v}^{\downarrow}$.*

PROOF. By induction with respect to $|\mathbf{u} \setminus \mathbf{u}^{\downarrow}|$. If $|\mathbf{u} \setminus \mathbf{u}^{\downarrow}| = 0$ then $\mathbf{u} = \mathbf{u}^{\downarrow}$ and there is nothing to prove. Assume $|\mathbf{u} \setminus \mathbf{u}^{\downarrow}| = n+1$, $n \in \mathbb{N}_0$. Let $(a,n) \in \mathbf{u} \setminus \mathbf{u}^{\downarrow}$. As $a \leq \bigoplus \mathbf{u} = \bigoplus \mathbf{v} \leq a'$, a occurs in $\mathbf{v}$. Suppose that a occurs in $\mathbf{v}^{\downarrow}$. Then $\iota(a).a \leq \bigoplus \mathbf{v}^{\downarrow} \perp a$, which is a contradiction. Thus, a occurs in $\mathbf{v} \setminus \mathbf{v}^{\downarrow}$, say $(a,k) \in \mathbf{v} \setminus \mathbf{v}^{\downarrow}$. Put $\mathbf{u}_1 = \mathbf{u} \setminus \{(a,n)\}$, $\mathbf{v}_1 = \mathbf{v} \setminus \{(a,k)\}$. We have $\bigoplus \mathbf{u}_1 = \bigoplus \mathbf{v}_1$, $\mathbf{u}^{\downarrow} = \mathbf{u}_1^{\downarrow}$, $\mathbf{v}^{\downarrow} = \mathbf{v}_1^{\downarrow}$. By induction hypothesis, $\bigoplus \mathbf{u}_1^{\downarrow} = \bigoplus \mathbf{v}_1$. This completes the proof.

Proposition 4.4 *Let E be a finite homogeneous effect algebra, let $\mathbf{u}$ be an event of $\Omega(E)$. Then $\bigoplus \mathbf{u}^{\downarrow} = \bigvee [0, \bigoplus \mathbf{u}] \cap E_S$.*

PROOF. Let $x \in [0, \bigoplus \mathbf{u}] \cap E_S$. Let $\mathbf{v}_1, \mathbf{v}_2$ be events of $\Omega(E)$ such that $\bigoplus \mathbf{v}_1 = x$, $\bigoplus \mathbf{v}_2 = (\bigoplus \mathbf{u}) \ominus x$. By Lemma 3.9, $\mathbf{v}_1 \perp \mathbf{v}_2$. By Proposition 4.3, $\bigoplus(\mathbf{v}_1 \dot{\cup} \mathbf{v}_2) = \bigoplus \mathbf{u}$ implies that $\bigoplus(\mathbf{v}_1 \dot{\cup} \mathbf{v}_2)^{\downarrow} = \bigoplus \mathbf{u}^{\downarrow}$. Since $\bigoplus \mathbf{v}_1 \in E_S$, $\mathbf{v}_1^{\downarrow} = \mathbf{v}_1$. Thus, $(\mathbf{v}_1 \dot{\cup} \mathbf{v}_2)^{\downarrow} = \mathbf{v}_1^{\downarrow} \dot{\cup} \mathbf{v}_2^{\downarrow}$. Therefore, $x = \bigoplus \mathbf{v}_1 = \bigoplus \mathbf{v}_1^{\downarrow} \leq \bigoplus \mathbf{u}^{\downarrow}$.

Remark. Z. Riečanová proved [22] a version of Proposition 4.4 for atomic lattice ordered effect algebras. See also [23], Theorem 2.2.

Corollary 4.5 *Let E be a finite homogeneous effect algebra, let $\mathbf{f}$ be an event of $\Omega(E)$. Then $\bigoplus \mathbf{f}^{\uparrow} = \bigvee([\bigoplus \mathbf{f}, 1] \cap E_S)$.*

PROOF. Let $\mathbf{t}$ be a test of $\Omega(E)$ such that $\mathbf{f} \subseteq \mathbf{t}$. Put $\mathbf{g} = \mathbf{t} \setminus \mathbf{f}^{\downarrow}$. Since $\bigoplus \mathbf{t} \setminus \mathbf{f}^{\uparrow} \in E_S$, $\mathbf{t} \setminus \mathbf{f}^{\uparrow} \subseteq \mathbf{g}^{\downarrow}$. Suppose that $\mathbf{t} \setminus \mathbf{f}^{\uparrow} \neq \mathbf{g}^{\downarrow}$. Let a

be an atom occurring in $\mathbf{g}^{\downarrow} \setminus (\mathbf{t} \setminus \mathbf{f}^{\uparrow})$. Since $\oplus \mathbf{g}^{\downarrow} \setminus (\mathbf{t} \setminus \mathbf{f}^{\uparrow}) \in E_S$,

$$\{(a,1),\dots,(a,\iota(a))\} \subseteq \bigoplus \mathbf{g}^{\downarrow} \setminus (\mathbf{t} \setminus \mathbf{f}^{\uparrow}) \subseteq \mathbf{f}^{\uparrow} \setminus \mathbf{f}.$$

By Lemma 4.1, a occurs in $\mathbf{f} \setminus \mathbf{f}^{\downarrow}$. This implies that $\iota(a).a \perp a$, which is a contradiction. Therefore, $\mathbf{t} \setminus \mathbf{f}^{\uparrow} = \mathbf{g}^{\downarrow}$.

Let $x \in [\oplus \mathbf{f}, 1] \cap E_S$. We have

$$x' \in [0, (\bigoplus \mathbf{f})'] \cap E_S = [0, \bigoplus \mathbf{g}] \cap E_S.$$

By Proposition 4.4, $x' \leq \oplus \mathbf{g}^{\downarrow}$, hence $x \geq \oplus \mathbf{g}^{\downarrow} = \oplus \mathbf{f}^{\uparrow}$.

Corollary 4.6 *Let E be a finite homogeneous effect algebra. Let $\mathbf{f}$ be an event of $\Omega(E)$. Then $\oplus \mathbf{f}^{\downarrow} \wedge \oplus \mathbf{f} \setminus \mathbf{f}^{\downarrow} = 0$*

PROOF. $\oplus \mathbf{f}^{\downarrow} \wedge \oplus \mathbf{f} \setminus \mathbf{f}^{\downarrow} \leq \oplus \mathbf{f}^{\downarrow} \wedge (\oplus \mathbf{f}^{\downarrow})' = 0$

Proposition 4.7 *Let E be a finite homogeneous effect algebra, let $\mathbf{f}, \mathbf{g}$ be events of $\Omega(E)$. Then $\mathbf{f} \sim \mathbf{g}$ iff $\mathbf{f} \setminus \mathbf{f}^{\downarrow} = \mathbf{g} \setminus \mathbf{g}^{\downarrow}$ and $\oplus \mathbf{f}^{\downarrow} = \oplus \mathbf{g}^{\downarrow}$.*

PROOF. Suppose that $\mathbf{f} \sim \mathbf{g}$. This implies that $\oplus \mathbf{f} = \oplus \mathbf{g}$. By Corollary 4.4,

$$\bigoplus \mathbf{f}^{\downarrow} = \bigvee([0, \bigoplus \mathbf{f}] \cap E_S) = \bigvee([0, \bigoplus \mathbf{g}] \cap E_S) = \bigoplus \mathbf{g}^{\downarrow}.$$

As $\mathbf{f} \sim \mathbf{g}$, there are tests $\mathbf{r}, \mathbf{t}$ of $\Omega(E)$ such that $\mathbf{r} \supseteq \mathbf{f}$, $\mathbf{t} \supseteq \mathbf{g}$ and $\mathbf{r} \setminus \mathbf{f} = \mathbf{t} \setminus \mathbf{g}$. Let $(a,n) \in \mathbf{f} \setminus \mathbf{f}^{\downarrow}$. By Lemma 4.1, a occurs in $\mathbf{f}^{\uparrow} \setminus \mathbf{f}$. As $\mathbf{f}^{\uparrow} \setminus \mathbf{f} \subseteq \mathbf{r} \setminus \mathbf{f} = \mathbf{t} \setminus \mathbf{g}$, a occurs in $\mathbf{t}$. Since $\mathbf{t} = \mathbf{t}^{\uparrow}$ and a occurs in $\mathbf{t}$, $(a,n) \in \mathbf{t}$. Suppose that $(a,n) \in \mathbf{t} \setminus \mathbf{g}$. Then $(a,n) \in \mathbf{r} \setminus \mathbf{f}$. However, this contradicts $(a,n) \in \mathbf{f} \setminus \mathbf{f}^{\downarrow} \subseteq \mathbf{f}$. Thus, $(a,n) \in \mathbf{g}$. Suppose that $(a,n) \in \mathbf{g}^{\downarrow}$. Then

$$\iota(a).a \leq \bigoplus \mathbf{g}^{\downarrow} = \bigoplus \mathbf{f}^{\downarrow} \perp \bigoplus(\mathbf{f} \setminus \mathbf{f}^{\downarrow}) \geq a.$$

This is a contradiction, hence $(a,n) \in \mathbf{g} \setminus \mathbf{g}^{\downarrow}$. We have proved that $\mathbf{f} \sim \mathbf{g}$ implies that $\mathbf{f} \setminus \mathbf{f}^{\downarrow} \subseteq \mathbf{g} \setminus \mathbf{g}^{\downarrow}$. By symmetry, we obtain $\mathbf{f} \setminus \mathbf{f}^{\downarrow} = \mathbf{g} \setminus \mathbf{g}^{\downarrow}$.

Suppose that $\mathbf{f}\backslash\mathbf{f}^{\downarrow} = \mathbf{g}\backslash\mathbf{g}^{\downarrow}$ and $\oplus\mathbf{f}^{\downarrow} = \oplus\mathbf{g}^{\downarrow}$. Since $\oplus(\mathbf{f}^{\uparrow}\backslash\mathbf{f}^{\downarrow}) \in E_S$, Proposition 4.2 implies that $\mathbf{f}^{\uparrow}\setminus\mathbf{f}^{\downarrow} = (\mathbf{f}^{\uparrow}\setminus\mathbf{f}^{\downarrow})^{\uparrow}$. By Lemma 4.1, an atom a occurs in $\mathbf{f}^{\uparrow}\setminus\mathbf{f}^{\downarrow}$ iff a occurs in $\mathbf{f}\setminus\mathbf{f}^{\downarrow} = \mathbf{g}\setminus\mathbf{g}^{\downarrow}$. Therefore, $\mathbf{f}^{\uparrow}\setminus\mathbf{f}^{\downarrow} = \mathbf{g}^{\uparrow}\setminus\mathbf{g}^{\downarrow}$. This implies that

$$\bigoplus\mathbf{f}^{\uparrow} = \bigoplus\mathbf{f}^{\downarrow}\oplus\bigoplus\mathbf{f}^{\uparrow}\setminus\mathbf{f}^{\downarrow} = \bigoplus\mathbf{g}^{\downarrow}\oplus\bigoplus\mathbf{g}^{\uparrow}\setminus\mathbf{g}^{\downarrow} = \bigoplus\mathbf{g}^{\uparrow}.$$

Let $\mathbf{h}$ be any test of $\Omega(E)$ such that $\oplus\mathbf{h} = (\oplus\mathbf{f}^{\uparrow})'$. By the sharpness of $\oplus\mathbf{h}$, we have $\mathbf{h}\cap\mathbf{f}^{\uparrow} = \mathbf{h}\cap\mathbf{g}^{\uparrow} = \emptyset$. By Lemma 3.8, $\mathbf{f}$ loc $((\mathbf{f}^{\uparrow}\setminus\mathbf{f})\dot\cup\mathbf{h})$ and $\mathbf{g}$ loc $((\mathbf{g}^{\uparrow}\setminus\mathbf{g})\dot\cup\mathbf{h})$. Thus, $\mathbf{f}\sim\mathbf{g}$.

Proposition 4.8 *Let E be a finite homogeneous effect algebra. $A\subseteq E$ is compatible in iff $\phi_E^{-1}(A)$ is compatible in $O(E)$*

PROOF. Suppose that $\phi_E^{-1}(A)$ is compatible. Since every homomorphism preserves compatible sets, $A\subseteq\phi_E(\phi_E^{-1}(A))$ is compatible.

Conversely, suppose that $A\subseteq E$ is compatible. Since E is homogeneous, there is a block $B\supseteq A$ of E. By Proposition 3.3, there is a unique atomic test $\mathbf{t}$ of E such that $B = \{\oplus\mathbf{f} : \mathbf{f}\leq\mathbf{t}\}$. Note that $B = \{\oplus\mathbf{f} : \mathbf{f}\subseteq\mathbf{t}_S\}$. We have $[\mathbf{g}]_\sim\in\phi_E^{-1}(B)$ iff there exists $\mathbf{f}\subseteq\mathbf{t}_S$ such that $\oplus\mathbf{g} = \oplus\mathbf{f}$. It follows from Proposition 4.3 that $\oplus\mathbf{g}^{\downarrow} = \oplus\mathbf{f}^{\downarrow}$. By Proposition 4.7, $\mathbf{g}^{\downarrow}\sim\mathbf{f}^{\downarrow}$. Moreover, by Proposition 4.3 a occurs in $\mathbf{f}\setminus\mathbf{f}^{\downarrow}$ iff a occurs in $\mathbf{g}\setminus\mathbf{g}^{\downarrow}$, hence $\mathbf{g}\setminus\mathbf{g}^{\downarrow}\subseteq\mathbf{t}_S$. We have

$$g = \mathbf{g}^{\downarrow}\dot\cup(\mathbf{g}\setminus\mathbf{g}^{\downarrow})\sim\mathbf{f}^{\downarrow}\dot\cup(\mathbf{g}\setminus\mathbf{g}^{\downarrow})\subseteq\mathbf{t}_S,$$

hence $[\mathbf{g}]_\sim$ is in $O(E)$ covered by the orthogonal word $C = ([\{(a,n)\}]_\sim)_{(a,n)\in\mathbf{t}_S}$. Thus, C is an orthogonal cover of $\phi_E^{-1}(B)\supseteq\phi_E^{-1}(A)$.

Corollary 4.9 *Let E be a finite homogeneous effect algebra, let B be a block of E. Then $\phi_E^{-1}(B)$ is a block of $O(E)$.*

PROOF. By Proposition 4.8, $\phi_E^{-1}(B)$ is compatible. Let $x\in O(E)$ be such that $\phi_E^{-1}(B)\cup\{x\}$ is compatible. Again, by Proposition 4.8, $\phi_E(\phi_E^{-1}(B)\cup\{x\}) = B\cup\{\phi_E(x)\}$ is compatible. Since

B is a maximal compatible subset of E, $\phi_E(x) \in B$. Therefore, $x \in \phi_E^{-1}(B)$ and $\phi_E^{-1}(B)$ is a maximal compatible subset of $O(E)$.

Corollary 4.10 *Let E be a finite homogeneous effect algebra, let B be a block of $O(E)$. Then $\phi_E(B)$ is a block of E.*

PROOF. By Proposition 4.8, $\phi_E(B)$ is compatible. Since E is homogeneous, there is a block $B_0 \supseteq \phi_E(B)$. By Corollary 4.9, $\phi_E^{-1}(B_0)$ is a block of $O(E)$. By the maximality of B, $B = \phi_E^{-1}(B_0)$ and we obtain $\phi_E(B) = \phi_E(\phi_E^{-1}(B_0)) = B_0$.

Proposition 4.11 *Let E be a finite homogeneous effect algebra, let $\mathbf{f}, \mathbf{g}$ be tests of $\Omega(E)$. Then $[\mathbf{g}]_\sim \leq [\mathbf{f}]_\sim$ iff $\bigoplus \mathbf{g}^\downarrow \dot\cup ((\mathbf{g} \setminus \mathbf{g}^\downarrow) \setminus (\mathbf{f} \setminus \mathbf{f}^\downarrow)) \leq \bigoplus \mathbf{f}^\downarrow$.*

PROOF.

$\Rightarrow$: Suppose $[\mathbf{g}]_\sim \leq [\mathbf{f}]_\sim$. There exists a test $\mathbf{h}$ such that $[\mathbf{g}]_\sim \oplus [\mathbf{h}]_\sim = [\mathbf{f}]_\sim$. By the algebraicity of $\Omega(E)$, $\mathbf{g} \perp \mathbf{h}$ and $\mathbf{g} \dot\cup \mathbf{h} \sim \mathbf{f}$. According to Proposition 4.7, $\mathbf{g} \dot\cup \mathbf{h} \sim \mathbf{f}$ implies that $\bigoplus \mathbf{f}^\downarrow = \bigoplus (\mathbf{g} \dot\cup \mathbf{h})^\downarrow$ and $\mathbf{f} \setminus \mathbf{f}^\downarrow = (\mathbf{g} \dot\cup \mathbf{h}) \setminus (\mathbf{g} \dot\cup \mathbf{h})^\downarrow$. Since

$$((\mathbf{g} \setminus \mathbf{g}^\downarrow) \setminus (\mathbf{f} \setminus \mathbf{f}^\downarrow)) \cap (\mathbf{f} \setminus \mathbf{f}^\downarrow) = ((\mathbf{g} \setminus \mathbf{g}^\downarrow) \setminus (\mathbf{f} \setminus \mathbf{f}^\downarrow)) \cap ((\mathbf{g} \dot\cup \mathbf{h}) \setminus (\mathbf{g} \dot\cup \mathbf{h})^\downarrow) = \emptyset,$$

we have

$$(\mathbf{g} \setminus \mathbf{g}^\downarrow) \setminus (\mathbf{f} \setminus \mathbf{f}^\downarrow) \subseteq (\mathbf{g} \dot\cup \mathbf{h})^\downarrow.$$

Moreover, $\mathbf{g}^\downarrow \subseteq (\mathbf{g} \dot\cup \mathbf{h})^\downarrow$ and $\mathbf{g}^\downarrow \perp (\mathbf{g} \setminus \mathbf{g}^\downarrow) \setminus (\mathbf{f} \setminus \mathbf{f}^\downarrow)$. Hence, $\mathbf{g}^\downarrow \dot\cup ((\mathbf{g} \setminus \mathbf{g}^\downarrow) \setminus (\mathbf{f} \setminus \mathbf{f}^\downarrow)) \subseteq (\mathbf{g} \dot\cup \mathbf{h})^\downarrow$ and $\bigoplus \mathbf{g}^\downarrow \dot\cup ((\mathbf{g} \setminus \mathbf{g}^\downarrow) \setminus (\mathbf{f} \setminus \mathbf{f}^\downarrow)) \leq \bigoplus (\mathbf{g} \dot\cup \mathbf{h})^\downarrow = \bigoplus \mathbf{f}^\downarrow$

$\Leftarrow$: Suppose that

$$\bigoplus \mathbf{g}^\downarrow \dot\cup ((\mathbf{g} \setminus \mathbf{g}^\downarrow) \setminus (\mathbf{f} \setminus \mathbf{f}^\downarrow)) \leq \bigoplus \mathbf{f}^\downarrow.$$

By Proposition 4.4 and Corollary 4.5 (a) and (b),

$$\bigoplus [\mathbf{g}^\downarrow \dot\cup ((\mathbf{g} \setminus \mathbf{g}^\downarrow) \setminus (\mathbf{f} \setminus \mathbf{f}^\downarrow))]^\uparrow \leq \bigoplus \mathbf{f}^\downarrow.$$

By definition of $^{\uparrow}$,

$$[\mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))]^{\uparrow}=\mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))^{\uparrow}.$$

Let us write

$$\mathbf{h}_1=((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))^{\uparrow}\setminus((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow})).$$

We then have

$$\mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))^{\uparrow}=\mathbf{g}^{\downarrow}\dot{\cup}\mathbf{h}_1\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow})).$$

Let $\mathbf{h}_2$ be any event of $\Omega(E)$ such that

$$\bigoplus\mathbf{h}_2=\bigoplus\mathbf{f}^{\downarrow}\ominus\bigoplus\mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))^{\uparrow}.$$

By Lemma 3.9,

$$\mathbf{h}_2\perp\mathbf{g}^{\downarrow}\dot{\cup}[((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))]^{\uparrow}.$$

By Proposition 4.7,

$$\bigoplus\mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))\dot{\cup}\mathbf{h}_1\dot{\cup}\mathbf{h}_2=\bigoplus\mathbf{f}^{\downarrow}\in E_S$$

implies that

$$\mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))\dot{\cup}\mathbf{h}_1\dot{\cup}\mathbf{h}_2\sim\mathbf{f}^{\downarrow}.$$

Since $\mathbf{f}^{\downarrow}\perp(\mathbf{f}\setminus\mathbf{f}^{\downarrow})$ and $\Omega(E)$ is algebraic, $(\mathbf{f}\setminus\mathbf{f}^{\downarrow})\perp\mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))\dot{\cup}\mathbf{h}_1\dot{\cup}\mathbf{h}_2$ and we have

$$(\mathbf{f}\setminus\mathbf{f}^{\downarrow})\dot{\cup}\mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))\dot{\cup}\mathbf{h}_1\dot{\cup}\mathbf{h}_2\sim\mathbf{f}^{\downarrow}\dot{\cup}(\mathbf{f}\setminus\mathbf{f}^{\downarrow})=\mathbf{f}.$$

Moreover,

$$\mathbf{g}=\mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\cap(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))\subseteq\mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))\dot{\cup}(\mathbf{f}\setminus\mathbf{f}^{\downarrow})$$

Thus $[\mathbf{g}]_{\sim}\leq[\mathbf{f}]_{\sim}$.

Proposition 4.12 *Let E be a finite homogeneous effect algebra. Let $\mathbf{f}$, $\mathbf{g}$ be events of $\Omega(E)$. Then $[\mathbf{g}]_{\sim}\leq[\mathbf{f}]_{\sim}$ if and only if $\oplus\mathbf{g}^{\downarrow}\leq\oplus\mathbf{f}^{\downarrow}$ and for all $(a,n)\in\mathbf{g}$ we have $a\leq\oplus\mathbf{f}^{\downarrow}$ or $(a,n)\in\mathbf{f}\setminus\mathbf{f}^{\downarrow}$.*

PROOF.

($\Rightarrow$): By Proposition 4.11.

($\Leftarrow$): Let $(a,n) \in \mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))$. If $(a,n)\in\mathbf{g}^{\downarrow}$, then $a \leq \oplus\,\mathbf{g}^{\downarrow} \leq \oplus\,\mathbf{f}^{\downarrow}$. If $(a,n) \in ((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))$, then $(a,n) \notin \mathbf{f}\setminus\mathbf{f}^{\downarrow}$ and hence $a \leq \oplus\,\mathbf{g}^{\downarrow}$. Therefore, for all $(a,n) \in \mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))$ we have $a \leq \oplus\,\mathbf{f}$.

Let $A = \{a_1,\ldots,a_n\}$ be the set of all atoms occurring in $\mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))$. Since A is a compatible set, there exists a block $B \supseteq A$. Since B is a finite MV-effect algebra, B is isomorphic to a direct product of chains generated by elements of $\mathrm{At}(B) \supseteq A$. This implies that $a_1 \oplus \ldots \oplus a_n = a_1 \vee \ldots \vee a_n$.

Consider now the event $\mathbf{a} = \{(a_1,1),\ldots,(a_n,1)\}$. By above part of the proof, $\oplus\,\mathbf{a} \leq \oplus\,\mathbf{f}^{\downarrow}$. Therefore, $\oplus\,\mathbf{a}^{\uparrow} \leq \oplus\,\mathbf{f}^{\downarrow}$. It remains to observe that $\oplus\,\mathbf{a}^{\uparrow} \geq \mathbf{g}^{\downarrow}\dot{\cup}((\mathbf{g}\setminus\mathbf{g}^{\downarrow})\setminus(\mathbf{f}\setminus\mathbf{f}^{\downarrow}))$.

Theorem 4.13 *For every finite lattice ordered effect algebra E, $O(E)$ is an orthomodular lattice.*

PROOF. Recall that $O(E)$ is an orthoalgebra. It remains to prove that $O(E)$ is lattice ordered. Let $\mathbf{f}$, $\mathbf{g}$ be events of $\Omega(E)$. Let $\mathbf{u}_1$ be any event of $\Omega(E)$ satisfying $\oplus\,\mathbf{u}_1 = (\oplus\,\mathbf{f}^{\downarrow}) \wedge (\oplus\,\mathbf{g}^{\downarrow})$. Let

$$\begin{aligned}
\mathbf{u}_2 &= (\mathbf{f}\setminus\mathbf{f}^{\downarrow}) \cap (\mathbf{g}\setminus\mathbf{g}^{\downarrow})\\
\mathbf{u}_3 &= \{(a,n) \in \mathbf{f}\setminus\mathbf{f}^{\downarrow} : a \leq \bigoplus \mathbf{g}^{\downarrow}\}\\
\mathbf{u}_4 &= \{(a,n) \in \mathbf{g}\setminus\mathbf{g}^{\downarrow} : a \leq \bigoplus \mathbf{f}^{\downarrow}\}
\end{aligned}$$

We shall prove that the $\mathbf{u}_i$'s are mutually disjoint and that $\mathbf{u}_1\dot{\cup}\mathbf{u}_2\dot{\cup}\mathbf{u}_3\dot{\cup}\mathbf{u}_4$ is an event of $\Omega(E)$ such that $[\mathbf{u}]_{\sim} = [\mathbf{f}]_{\sim} \wedge [\mathbf{g}]_{\sim}$. By Corollary 4.6, it is easy to check that for every $i \neq j$, $\mathbf{u}_i \wedge \mathbf{u}_j = 0$ and hence $\mathbf{u}_i \cap \mathbf{u}_j = \emptyset$. Moreover, $\oplus\,\mathbf{u}_1 \perp \oplus\,\mathbf{u}_2$ and, since $\oplus\,\mathbf{u}_1 \wedge \oplus\,\mathbf{u}_2 = 0$, $\oplus\,\mathbf{u}_1 \vee \oplus\,\mathbf{u}_2 = \oplus\,\mathbf{u}_1 \oplus \oplus\,\mathbf{u}_2$. Similarly, $\oplus\,\mathbf{u}_3 \perp \oplus\,\mathbf{u}_4$ and $\oplus\,\mathbf{u}_3 \vee \oplus\,\mathbf{u}_4 = \oplus\,\mathbf{u}_3 \oplus \oplus\,\mathbf{u}_4$. Since $\oplus\,\mathbf{u}_1 \leq (\oplus\,\mathbf{u}_3)'$ and $\oplus\,\mathbf{u}_2 \leq (\oplus\,\mathbf{u}_3)'$, $\oplus\,\mathbf{u}_1 \oplus \oplus\,\mathbf{u}_2 = \oplus\,\mathbf{u}_1 \vee \oplus\,\mathbf{u}_2 \leq (\oplus\,\mathbf{u}_3)'$. Similarly,

$\bigoplus \mathbf{u}_1 \oplus \bigoplus \mathbf{u}_2 \leq (\bigoplus \mathbf{u}_4)'$, hence

$$\bigoplus \mathbf{u}_1 \oplus \bigoplus \mathbf{u}_2 \leq (\bigoplus \mathbf{u}_3)' \wedge (\bigoplus \mathbf{u}_4)' = (\bigoplus \mathbf{u}_3 \vee \bigoplus \mathbf{u}_4)' = (\bigoplus \mathbf{u}_3 \oplus \bigoplus \mathbf{u}_4)'.$$

Thus, $\bigoplus \mathbf{u}_1 \oplus \bigoplus \mathbf{u}_2 \oplus \bigoplus \mathbf{u}_3 \oplus \bigoplus \mathbf{u}_4$ exists and, by Lemma 3.7, $\mathbf{u}_1 \dot\cup \mathbf{u}_2 \dot\cup \mathbf{u}_3 \dot\cup \mathbf{u}_4$ is an event of $\Omega(E)$.

It is obvious that $\mathbf{u}_1 \subseteq \mathbf{u}^{\downarrow}$. Assume that $\mathbf{u}^{\downarrow} \neq \mathbf{u}_1$. Then there exists an atom a such that $A = \{(a,1), \ldots, (a, \iota(a))\} \subseteq \mathbf{u}_2 \dot\cup \mathbf{u}_3 \dot\cup \mathbf{u}_4$. Suppose that $A \subseteq \mathbf{u}_4$. Since $\mathbf{u}_4 \subseteq \mathbf{g} \setminus \mathbf{g}^{\downarrow}$, $0 < \bigoplus A \leq \bigoplus \mathbf{g} \setminus \mathbf{g}^{\downarrow}$. As $\bigoplus A \in E_S$, this implies that $\bigoplus \mathbf{g}^{\downarrow} < \bigoplus \mathbf{g}^{\downarrow} \oplus \bigoplus A \in E_S \cap [0, \bigoplus \mathbf{g}]$. By Corollary 4.4, this implies that $\bigoplus A = 0$; a contradiction. Similarly, the assumption $A \subseteq \mathbf{u}_2 \dot\cup \mathbf{u}_3$ leads to a contradiction. Therefore, there exist n, m such that $(a,n) \in \mathbf{u}_2 \dot\cup \mathbf{u}_3$ and $(a,m) \in \mathbf{u}_4$. However, this is a contradiction with $\bigoplus(\mathbf{f} \setminus \mathbf{f}^{\downarrow}) \wedge \bigoplus \mathbf{f}^{\downarrow} = 0$, and we have proved that $\mathbf{u}^{\downarrow} = \mathbf{u}_1$. Thus, $\bigoplus \mathbf{u}^{\downarrow} \leq \bigoplus \mathbf{f}^{\downarrow}, \bigoplus \mathbf{g}^{\downarrow}$ and now it follows from Proposition 4.12 that $[\mathbf{u}]_\sim \leq [\mathbf{f}]_\sim, [\mathbf{g}]_\sim$.

Let $\mathbf{v}$ be an event of $\Omega(E)$ such that $[\mathbf{v}]_\sim \leq [\mathbf{f}]_\sim, [\mathbf{g}]_\sim$. We shall prove that $[\mathbf{v}]_\sim \leq [\mathbf{u}]_\sim$. Since $\bigoplus \mathbf{v}^{\downarrow} \leq \bigoplus \mathbf{f}^{\downarrow}, \bigoplus \mathbf{g}^{\downarrow}$, we have

$$\bigoplus \mathbf{v}^{\downarrow} \leq \bigoplus \mathbf{f}^{\downarrow} \wedge \bigoplus \mathbf{g}^{\downarrow} = \bigoplus \mathbf{u}_1 = \bigoplus \mathbf{u}^{\downarrow}.$$

Let $(a,n) \in \mathbf{v}$. By Proposition 4.12, $[\mathbf{v}]_\sim \leq [\mathbf{f}]_\sim$ implies that $a \leq \bigoplus \mathbf{f}^{\downarrow}$ or $a \in \mathbf{f} \setminus \mathbf{f}^{\downarrow}$ and $[\mathbf{v}]_\sim \leq [\mathbf{g}]_\sim$ implies that $a \leq \bigoplus \mathbf{g}^{\downarrow}$ or $a \in \mathbf{g} \setminus \mathbf{g}^{\downarrow}$.

Thus, at least one of the following conditions must be satisfied.

(A) $a \leq \bigoplus \mathbf{f}^{\downarrow}$ and $a \leq \bigoplus \mathbf{g}^{\downarrow}$.
(B) $a \leq \bigoplus \mathbf{f}^{\downarrow}$ and $(a,n) \in \mathbf{g} \setminus \mathbf{g}^{\downarrow}$.
(C) $(a,n) \in \mathbf{f} \setminus \mathbf{f}^{\downarrow}$ and $a \leq \bigoplus \mathbf{g}^{\downarrow}$.
(D) $(a,n) \in \mathbf{f} \setminus \mathbf{f}^{\downarrow}$ and $a \leq \bigoplus \mathbf{g}^{\downarrow}$.

Now we see that

(A) implies that $a \leq \bigoplus \mathbf{f}^{\downarrow} \wedge \bigoplus \mathbf{g}^{\downarrow} \leq \bigoplus \mathbf{u}^{\downarrow}$,
(B) implies that $(a,n) \in \mathbf{u}_4$,
(C) implies that $(a,n) \in \mathbf{u}_3$, and
(D) implies that $(a,n) \in \mathbf{u}_2$.

As $\mathbf{u} \setminus \mathbf{u}^{\downarrow} = \mathbf{u}_2 \dot{\cup} \mathbf{u}_3 \dot{\cup} \mathbf{u}_4$, we see that $a \leq \bigoplus \mathbf{u}^{\downarrow}$ or $(a, n) \in \mathbf{u} \setminus \mathbf{u}^{\downarrow}$. By Proposition 4.12, $[\mathbf{v}]_{\sim} \leq [\mathbf{u}]_{\sim}$.

References

[1] D. Foulis, M. Bennett, Effect algebras and unsharp quantum logics, Found. Phys. 24 (1994) 1331–1352.

[2] F. Kôpka, F. Chovanec, D-posets, Math. Slovaca 44 (1994) 21–34.

[3] R. Giuntini, H. Greuling, Toward a formal language for unsharp properties, Found. Phys. 19 (1994) 769–780.

[4] G. Ludwig, Foundations of Quantum Mechanics, Springer-Verlag, 1983.

[5] P. Bush, M. Grabowski, P. Lahti, Operational Quantum Physics, Springer-Verlag, 1995.

[6] D. Foulis, R. Greechie, G. Rütimann, Filters and supports in othoalgebras, Int. J. Theor. Phys. 35 (1995) 789–802.

[7] C. Chang, Algebraic analysis of many-valued logics, Trans. Amer. Math. Soc. 89 (1959) 74–80.

[8] F. Chovanec, F. Kôpka, Boolean D-posets, Tatra Mt. Math. Publ 10 (1997) 1–15.

[9] M. Bennett, D. Foulis, Phi-symmetric effect algebras, Foundations of Physics 25 (1995) 1699–1722.

[10] P. Pták, S. Pulmannová, Orthomodular Structures as Quantum Logics, Kluwer, Dordrecht, 1991.

[11] G. Kalmbach, Orthomodular Lattices, Academic Press, New York, 1983.

[12] L. Beran, Orthomodular Lattices (Algebraic Approach), Dordrecht, 1985.

[13] Z. Riečanová, A generalization of blocks for lattice effect algebras, Int. J. Theor. Phys. 39 (2000) 855–865.

[14] G. Jenča, Z. Riečanová, On sharp elements in lattice ordered effect algebras, BUSEFAL 80 (1999) 24–29.

[15] G. Jenča, Blocks of homogeneous effect algebras, Bull. Austr. Math. Soc. 64 (2001) 81–98.

[16] K. Ravindran, On a structure theory of effect algebras, Ph.D. thesis, Kansas State Univ., Manhattan, Kansas (1996).

[17] G. Jenča, S. Pulmannová, Quotients of partial abelian monoids and the Riesz decomposition property, Algebra univ. (to appear) .

[18] A. Dvureenskij, S. Pulmannov, New Trends in Quantum Structures, Kluwer, Dordrecht and Ister Science, Bratislava, 2000.

[19] S. Gudder, Effect test spaces, Intern. J. Theor. Phys. 36 (1997) 2681–2705.

[20] D. Foulis, C. Randall, Operational quantum statistics. I. Basic concepts, J. Math. Phys. 13 (1972) 1667–1675.

[21] D. Foulis, M. Bennett, Tensor products of orthoalgebras, Order 10 (1993) 271–282.

[22] Z. Riečanová, personal communication (2002).

[23] Z. Riečanová, Distributive atomic effect algebras, preprint, FEI STU Bratislava (2002).